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Rearrangement Estimates for Fourier Transforms in L^p and H^p in Terms of Moduli of Continuity

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Abstract. One of the main purposes of this paper is to obtain estimates for Fourier transforms of functions in $L^p(\mathbb{R}^n)$ $(1 \le p \le 2)$ in terms of their moduli of continuity. More precisely, we study the following problem: find sharp conditions on the modulus of continuity of a function $f \in L^p(\mathbb{R}^n)$, under which the non–increasing rearragement of $\hat{f}$, the Fourier transform of f, is integrable against a given non–negative weight function ρ. We shall also study similar problems for the Fourier transforms of functions or distributions in the Hardy spaces $H^p(\mathbb{R}^n)$ $(0 < p \le 1,\ n \in \mathbb{N})$.

1. Introduction

One of the important questions in the theory of Fourier series is the following: How are the smoothness properties of a function reflected on the behaviour of its Fourier coefficients? In 1914 S. N. Bernstein proved that for each function $f \in \mathrm{Lip}\,\alpha$ $(\alpha > 1/2)$, the sequence of its Fourier coefficients belongs to ℓ^1, and this property fails for $\alpha = 1/2$. Later on, more general estimates of Fourier coefficients were obtained by S. N. Bernstein, H. Weyl, O. Szasz, G. G. Lorentz, S. B. Stechkin, A. A. Konjushkov, A. Pietsch and other authors (see [B], [Z] and [Pi]). We note that in the process to obtain those estimates, the crucial rôle was played by the Hausdorff–Young and Hardy–Littlewood inequalities.

Our goal is to obtain estimates for the rearrangements of Fourier transforms. We start by recalling some definitions and known results.

We shall denote by $S_0(\mathbb{R}^n)$ the class of all measurable functions f on $\mathbb{R}^n$, which are finite almost everywhere and satisfy the condition

$$(1.1) \qquad \lambda_f(y) \equiv |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty \quad \text{for all} \quad y > 0\,.$$

If a function f belongs to $S_0(\mathbb{R}^n)$, then its non–increasing rearrangement is defined to be the function f^* which is non–increasing on $]0,\infty[$ and is also equimeasurable with

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$|f(x)|$. Sometimes the functions in $S_0(\mathbb{R}^n)$ are called rearrangeable. Set

$$f^{\star\star}(t) \;=\; \frac{1}{t}\int_0^t f^\star(u)\,\mathrm{d}u\,, \quad t \;>\; 0\,.$$

For every $t > 0$ we have (see [B-S, p. 53])

$$f^{\star\star}(t) \;=\; \sup_{|E|=t}\frac{1}{t}\int_E |f(x)|\,\mathrm{d}x\,. \tag{1.2}$$

Besides, by the Hardy–Littlewood theorem (see [B-S, p. 44]), for every $f, g \in S_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x \;\le\; \int_0^\infty f^\star(t)g^\star(t)\,\mathrm{d}t\,. \tag{1.3}$$

Let $0 < p, r < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{p,r}(\mathbb{R}^n)$ if

$$\|f\|_{p,r} \;\equiv\; \left(\int_0^\infty \left(t^{1/p}f^\star(t)\right)^r \frac{\mathrm{d}t}{t}\right)^{1/r} \;<\; \infty\,.$$

The Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) \;=\; \int_{\mathbb{R}^n} f(x)e^{-2\pi \imath x\cdot\xi}\,\mathrm{d}x\,.$$

If $f \in L^p(\mathbb{R}^n)$ $(1 < p \le 2)$, then its Fourier transform satisfies the Hardy–Littlewood inequality, namely (see [B-L] and [Z])

$$\left(\int_{\mathbb{R}^n} |\xi|^{n(p-2)}\left|\hat{f}(\xi)\right|^p \mathrm{d}\xi\right)^{1/p} \;\le\; C\,\|f\|_p\,. \tag{1.4}$$

A stronger inequality is given by the Hardy–Littlewood–Paley theorem, which says that for each $f \in L^p(\mathbb{R}^n)$ $(1 < p \le 2)$

$$\left(\int_0^\infty t^{p-2}\hat{f}^\star(t)^p\,\mathrm{d}t\right)^{1/p} \;\le\; C\,\|f\|_p\,. \tag{1.5}$$

Furthermore, if $f \in L^{p,r}(\mathbb{R}^n)$ $(1 < p \le 2,\ r > 0)$, then

$$\left\|\hat{f}\right\|_{p',r} \;\le\; C\,\|f\|_{p,r}\,, \tag{1.6}$$

(see [H], [J-S 1] and [St]), where, as usual, p' denotes the exponent conjugate to p, given by $1/p + 1/p' = 1$. Inequality (1.5) is a particular case of (1.6) for $r = p$.

If we apply these inequalities to the finite differences or to the derivatives of a given function, we can get estimates for the Fourier transform of the function in terms of the L^p–norm $(1 < p \le 2)$ of the corresponding differential characteristic. In particular, taking into account that

$$\left|\hat{f}(\xi)\right| \;\asymp\; |\xi|^{-r}\sum_{k=1}^n \left|\widehat{D_k^r f}(\xi)\right| \quad (r \in \mathbb{N})\,, \tag{1.7}$$

we immediately obtain from (1.5) that for each function $f \in W_p^r(\mathbb{R}^n)$, ($n, r \in \mathbb{N}$, $1 < p \leq 2$)

$$\left(\int_0^\infty t^{pr/n+p-2} \hat{f}^\star(t)^p \,\mathrm{d}t\right)^{1/p} \leq C \sum_{k=1}^n \|D_k^r\|_p \,. \tag{1.8}$$

The notation $W_p^r(\mathbb{R}^n)$ ($1 \leq p < \infty$, $r \in \mathbb{N}$) stands for the Sobolev space of all functions $f \in L^p(\mathbb{R}^n)$ for which every weak derivative $D^s f$, $s = (s_1, \ldots, s_n)$, of order $|s| = s_1 + \cdots + s_n \leq r$ exists and belongs to $L^p(\mathbb{R}^n)$. The right–hand side of (1.8) contains only the norms of non–mixed derivatives. But it is well–known that, for $1 < p < \infty$

$$\sum_{|s|=r} \|D^s f\|_p \leq C \sum_{k=1}^n \|D_k^r f\|_p \,. \tag{1.9}$$

However (1.9) fails for $p = 1$ (see [B-I-N]).

For $p = 1$ the inequality (1.4) does not hold and it is impossible to use the method described above. In fact, for the Fourier transforms of functions in $L^1(\mathbb{R})$, the only available estimates are those based upon the obvious inequality $\|\hat{f}\|_\infty \leq \|f\|_1$.

Nevertheless, the situation is quite different in the multi–dimensional case. Namely, the following theorem was proved in [Bo1] and [P-W]. See also [Bo2].

Theorem 1.1. *If $f \in W_1^r(\mathbb{R}^n)$ ($n \geq 2$, $r \in \mathbb{N}$), then*

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|\, |\xi|^{r-n} \,\mathrm{d}\xi \leq C \sum_{|s|=r} \|D^s f\|_1 \,. \tag{1.10}$$

Contrary to what happened in (1.8), the right–hand side of (1.10) contains all the derivatives of order r. It was proved in [Ko2] that the norms of the mixed derivatives can be omitted.

Denote by $\widetilde{W}_p^r(\mathbb{R}^n)$ ($1 \leq p < \infty$, $r \in \mathbb{N}$) the space of all functions $f \in L^p(\mathbb{R}^n)$, for which every weak derivative $D_k^r f \equiv \partial^r f/\partial x_k^r$ exists and belongs to $L^p(\mathbb{R}^n)$. By (1.9) we have $\widetilde{W}_p^r(\mathbb{R}^n) = W_p^r(\mathbb{R}^n)$ for $1 < p < \infty$, but this is not true for $p = 1$.

The following result was obtained in [Ko2].

Theorem 1.2. *Let $f \in \widetilde{W}_1^r(\mathbb{R}^n)$ ($n \geq 2$, $r \in \mathbb{N}$). Then*

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|\, |\xi|^{r-n} \,\mathrm{d}\xi \leq C \sum_{k=1}^n \|D_k^r f\|_1 \tag{1.11}$$

and

$$\int_0^\infty \hat{f}^\star(t) t^{r/n-1} \,\mathrm{d}t \leq C \sum_{k=1}^n \|D_k^r f\|_1 \,. \tag{1.12}$$

It follows from (1.3) that, for $r \leq n$, the inequality (1.12) is stronger than (1.11). On the contrary, for $r > n$, (1.11) implies (1.12).

Of course, (1.11) and (1.12) fail to hold for $n = 1$. In particular, it is well–known that there exists $f \in W_1^1(\mathbb{R})$, such that $\hat{f} \notin L^1(\mathbb{R})$.

One of the main purposes of this paper is to obtain estimates for Fourier transforms of functions in $L^p(\mathbb{R}^n)$ $(1 \le p \le 2)$ in terms of their moduli of continuity. More precisely, we study the following problem: find sharp conditions on the modulus of continuity of a function $f \in L^p(\mathbb{R}^n)$, under which

$$\int_0^\infty \hat{f}^\star(t)^p \rho(t)\,\mathrm{d}t \;<\; \infty\,, \tag{1.13}$$

where ρ is a non–negative weight function. In other words, we ask how the inequality (1.5) improves if we put some additional conditions on the smoothness of f. As was mentioned above, for $1 < p \le 2$, we will use the inequality (1.5). For $p = 1$ and $n \ge 2$, the crucial rôle will be played by the inequality (1.12). We must point out that the results obtained do not extend to the case $n = p = 1$.

We shall also study similar problems for the Fourier transforms of functions or distributions in the Hardy spaces $H^p(\mathbb{R}^n)$ $(0 < p \le 1,\ n \in \mathbb{N})$.

It is well–known that for every $f \in H^p(\mathbb{R}^n)$ $(0 < p \le 1)$, its Fourier transform $\hat{f}$ is a continuous function on $\mathbb{R}^n$ satisfying the inequalities

$$\left|\hat{f}(\xi)\right| \;\le\; C\,|\xi|^{n(1/p-1)}\,\|f\|_{H^p}\,, \tag{1.14}$$

$$\left(\int_{\mathbb{R}^n} |\xi|^{n(p-2)}\left|\hat{f}(\xi)\right|^p \mathrm{d}\xi\right)^{1/p} \;\le\; C\,\|f\|_{H^p} \tag{1.15}$$

(see [F-S], [G-R] and [T-W]). Of course, the inequality (1.15) cannot have a "rearrangement" counterpart of the form (1.5), since $\hat{f}$ need not belong to $S_0(\mathbb{R}^n)$. Thus, our starting point will be other type of rearrangement inequality, which can be interpreted in terms of the Riesz potentials. It follows easily from (1.14) that for each $f \in H^p(\mathbb{R}^n)$ $(0 < p \le 1)$, the function

$$F(\xi) \;=\; |\xi|^{n(1-1/p)}\left|\hat{f}(\xi)\right|$$

belongs to the class $S_0(\mathbb{R}^n)$ (see the definition in (1.1)). Thus, instead of (1.13), we study the convergence of the integral

$$\int_0^\infty F^\star(t)^p \rho(t)\,\mathrm{d}t\,. \tag{1.16}$$

The main results of this paper are Theorems 2.5, 2.7 and 3.6. These theorems give necessary and sufficient conditions for the convergence of the integral in (1.13) or, respectively (1.16), for each f belonging to a class of functions having a given majorant for their L^p– (or, respectively H^p–) moduli of continuity.

In view of (1.3), the most interesting case is when the weight function ρ is decreasing.

We should mention the paper [J-S 2], in which some weighted rearrangement inequalities for the Fourier transform were studied.

2. Fourier transforms of functions in $L^p(\mathbb{R}^n)$

Let f be a function defined on $\mathbb{R}^n$. If $r \in \mathbb{N}$ and $1 \le j \le n$, we set

$$\Delta_j^r(h)f(x) \;=\; \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f(x+khe_j)\,, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}$, and e_j is the j–th unit vector from the canonical basis of $\mathbb{R}^n$.

If the function $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, then its L^p–modulus of continuity of order r with respect to the j–th variable is defined by

$$\omega_j^r(f;\delta)_p \;=\; \sup_{0\le h\le\delta}\left\|\Delta_j^r(h)f\right\|_p \quad (0 \le \delta < \infty)\,.$$

We also set

$$\omega^r(f;\delta)_p \;=\; \sum_{j=1}^{n}\omega_j^r(f;\delta)_p\,.$$

It is easy to see [N, p. 147], that for every $\nu \in \mathbb{N}$,

$$\omega_j^r(f;\nu\delta)_p \;\le\; \nu^r\omega_j^r(f;\delta)_p \quad (1 \le j \le n)\,. \tag{2.2}$$

If there exists a weak derivative $D_j^r f \in L^p(\mathbb{R}^n)$, then

$$\omega_j^r(f;\delta)_p \;\le\; \delta^r\left\|D_j^r f\right\|_p \quad (j = 1,\dots,n)\,. \tag{2.3}$$

Furthermore, for $f \in L^p(\mathbb{R}^n)$, $r \in \mathbb{N}$, $h > 0$ and $1 \le j \le n$, we set

$$S_{h,j}^r f(x) \;=\; h^{-r}\int_{[0,h]^r}\sum_{k=1}^{r}(-1)^{k-1}\binom{r}{k} f(x+k(u_1+\cdots+u_r)e_j)\,\mathrm{d}u$$

(see [B-S, p. 340]). It is easy to see that

$$\left\|S_{h,j}^r f\right\|_p \;\le\; 2^r\left\|f\right\|_p\,, \quad \left\|f - S_{h,j}^r f\right\|_p \;\le\; \omega_j^r(f;rh)_p\,. \tag{2.4}$$

Besides, there exists a weak derivative $D_j^r\big(S_{h,j}^r f\big)$ and

$$\left\|D_j^r\big(S_{h,j}^r f\big)\right\|_p \;\le\; Ch^{-r}\omega_j^r(f;h)_p\,. \tag{2.5}$$

Now, if we set

$$f_h(x) \;=\; S_{h,1}^r \dots S_{h,n}^r f(x)\,, \tag{2.6}$$

then it follows from (2.4) and (2.5) that

$$\|f - f_h\|_p \;\le\; C\omega^r(f;h)_p \tag{2.7}$$

and

$$\left\|D_j^r f_h\right\|_p \;\le\; Ch^{-r}\omega_j^r(f;h)_p \quad (j = 1,\dots,n)\,. \tag{2.8}$$

In this section we shall search for estimates of rearrangements of Fourier transforms of functions in $L^p(\mathbb{R}^n)$ $(1 \le p \le 2)$, in terms of their moduli of continuity. We begin with the case $1 < p \le 2$, which is simpler and can be analyzed completely by means of the inequality (1.5).

Proposition 2.1. *Let $f \in L^p(\mathbb{R}^n)$ $(1 < p \le 2,\ n \in \mathbb{N})$ and $r \in \mathbb{N}$. Then, for every $T > 0$*

$$\left(\int_T^\infty t^{p-2} \hat{f}^\star(t)^p \,\mathrm{d}t\right)^{1/p} \le C\omega^r\big(f; T^{-1/n}\big)_p \tag{2.9}$$

and

$$\left(\int_0^T t^{pr/n+p-2} \hat{f}^\star(t)^p \,\mathrm{d}t\right)^{1/p} \le CT^{r/n}\omega^r\big(f; T^{-1/n}\big)_p\,, \tag{2.10}$$

with a constant C depending only on p, n and r.

Proof. For $\tau > 0$ and $j = 1, \ldots, n$ set

$$\varphi_{\tau,j}(x) = \Delta_j^r(\tau) f(x)\,.$$

Then

$$\widehat{\varphi_{\tau,j}}(\xi) = \big(e^{2\pi\imath\tau\xi_j} - 1\big)^r \hat{f}(\xi)\,. \tag{2.11}$$

For $T \le t < \infty$ denote

$$E_t = \Big\{\xi \in \mathbb{R}^n \,:\, \big|\hat{f}(\xi)\big| \ge \hat{f}^\star(t)\Big\}\,.$$

Then $|E_t| \ge t$. There exists $j \equiv j(t)$ such that the set

$$E_t' \equiv \Big\{\xi \in E_t \,:\, \big|\xi_j\big| \ge T^{1/n}/2\Big\}$$

has measure $|E_t'| \ge |E_t|/(2n)$.

Let $h = T^{-1/n}$. For every $\xi \in E_t'$ we have

$$\begin{aligned}
\int_0^{rh} \big|e^{2\pi\imath\xi_j\tau} - 1\big|^r \,\mathrm{d}\tau &\ge \int_0^{rh} \big(1 - \cos 2\pi\xi_j\tau\big)^r \,\mathrm{d}\tau \\
&\ge \int_0^{rh} \big(1 - r\cos 2\pi\xi_j\tau\big) \,\mathrm{d}\tau \\
&\ge rh\left(1 - \frac{1}{2\pi h\,|\xi_j|}\right) \\
&\ge \frac{rh}{2}\,.
\end{aligned}$$

Thus, for every $\xi \in E_t'$ (see (2.11))

$$\hat{f}^\star(t) \le \big|\hat{f}(\xi)\big| \le \frac{2}{rh}\int_0^{rh} |\widehat{\varphi_{\tau,j}}(\xi)| \,\mathrm{d}\tau\,,$$

and we have

$$\hat{f}^\star(t) \le \frac{4n}{rh\,|E_t|}\int_0^{rh} \mathrm{d}\tau \int_{E_t'} |\widehat{\varphi_{\tau,j}}(\xi)| \,\mathrm{d}\xi \le \frac{4n}{rh}\int_0^{rh} \widehat{\varphi_{\tau,j}}^{\star\star}(t) \,\mathrm{d}\tau\,, \quad j = j(t)\,.$$

By applying (1.5) and Hardy's inequality ([B-S, p. 124]), we obtain

$$\begin{aligned}\left(\int_T^\infty t^{p-2}\hat f^\star(t)^p\,\mathrm{d}t\right)^{1/p} &\le C\sum_{j=1}^n\left(\frac1h\int_0^{rh}\mathrm{d}\tau\int_0^\infty \widehat{\varphi_{\tau,j}}^{\star\star}(t)^p t^{p-2}\,\mathrm{d}t\right)^{1/p}\\ &\le C\sum_{j=1}^n\left(\frac1h\int_0^{rh}\|\varphi_{\tau,j}\|_p^p\,\mathrm{d}\tau\right)^{1/p}\\ &\le C\omega^r(f;h)_p\,.\end{aligned}$$

Thus, we have proved (2.9).

Now, let f_h $\left(h = T^{-1/n}\right)$ be the function defined by (2.6). Then, by applying inequalities (1.5) and (1.8) to $f - f_h$ and f_h respectively and using (2.7) and (2.8), we obtain (with $\alpha = r/n + 1/p'$)

$$\begin{aligned}\left(\int_0^T t^{\alpha p-1}\hat f^\star(t)^p\,\mathrm{d}t\right)^{1/p} &\le T^{r/n}\left(\int_0^\infty t^{p-2}\widehat{f-f_h}^\star\left(\frac t2\right)^p\mathrm{d}t\right)^{1/p}\\ &\quad+\left(\int_0^\infty t^{\alpha p-1}\hat f_h^\star\left(\frac t2\right)^p\mathrm{d}t\right)^{1/p}\\ &\le C\left(h^{-r}\|f-f_h\|_p+\sum_{j=1}^n\left\|D_j^r f_h\right\|_p\right)\\ &\le Ch^{-r}\omega^r(f;h)_p\,.\end{aligned}$$

Now the proof is complete. □

Corollary 2.2. *If $f \in L^p(\mathbb{R}^n)$ $(1 < p \le 2,\ n \in \mathbb{N})$ and $r \in \mathbb{N}$, then*

$$\hat f^\star(t) \le Ct^{1/p-1}\omega^r(f;t^{-1/n})_p\,. \tag{2.12}$$

Now, let us consider the case $p = 1$. We first observe that the inequality (2.12) holds in this case too. Indeed, let $f \in L^1(\mathbb{R}^n)$. For each $\xi \in \mathbb{R}^n$, there exists j such that $|\xi_j| \ge |\xi|/n$. If we use (2.11) with $\tau = 1/\left(2|\xi_j|\right)$, we get

$$\left|\hat f(\xi)\right| = 2^{-r}\left|\widehat{\varphi_{\tau,j}}(\xi)\right| \le 2^{-r}\int_{\mathbb{R}^n}|\varphi_{\tau,j}(x)|\,\mathrm{d}x \le 2^{-r}\omega^r\left(f;\frac{n}{2|\xi|}\right)_1.$$

From here it immediately follows that

$$\hat f^\star(t) \le C_{n,r}\omega^r\left(f;t^{-1/n}\right)_1. \tag{2.13}$$

Let $1 \le p,\theta < \infty$, $\alpha > 0$ and $r > \alpha$ $(r \in \mathbb{N})$. The Besov space $B^\alpha_{p,\theta}(\mathbb{R}^n)$ consists of all those functions $f \in L^p(\mathbb{R}^n)$ for which

$$\|f\|_{b^\alpha_{p,\theta}} \equiv \left(\int_0^\infty\left(t^{-\alpha}\omega^r(f;t)_p\right)^\theta\frac{\mathrm{d}t}{t}\right)^{1/\theta} < \infty\,.$$

If we choose different integers $r > \alpha$, we obtain equivalent seminorms (see [B-I-N]).

From the inequalities (2.12) and (2.13), we immediately get

Corollary 2.3. *Let* $1 \le p \le 2,\ 1 \le \theta < \infty,\ \alpha > 0$ *and* $q = (\alpha/n + 1/p')^{-1}$. *If* $f \in B^{\alpha}_{p,\theta}(\mathbb{R}^n)$, *then*

$$\|\hat{f}\|_{L^{q,\theta}} \ \le\ \|f\|_{b^{\alpha}_{p,\theta}}\,.$$

This result was obtained by other methods by A. Pietsch (see [Pi, p. 270]). Setting $\theta = 1$ and $\alpha = n/p$, we obtain the Bernstein–Szasz theorem: if $f \in B^{n/p}_{p,1}(\mathbb{R}^n)$ $(1 \le p \le 2)$, then $\hat{f} \in L^1(\mathbb{R}^n)$ (see [Sz] and [Z]).

Note that the estimate (2.13) is sharp in the case $n = 1$ (see Proposition 2.8 below). But for $n \ge 2$ it can be strengthened.

Proposition 2.4. *Let* $f \in L^1(\mathbb{R}^n)$ $(n \ge 2)$ *and* $r \in \mathbb{N}$. *Then, for each* $T > 0$

$$\int_0^T t^{r/n-1}\hat{f}^{\star}(t)\,\mathrm{d}t \ \le\ C_{n,r}T^{r/n}\omega^r\big(f;T^{-1/n}\big)_1\,. \tag{2.14}$$

Proof. Let $h = T^{-1/n}$. For every $t > 0$, we have

$$\hat{f}^{\star}(t) \ \le\ \widehat{f_h}^{\star}(t) + \big\|\widehat{f - f_h}\big\|_{\infty}$$

(where f_h is defined by (2.6)). Furthermore by (2.7)

$$\big\|\widehat{f - f_h}\big\|_{\infty} \ \le\ \|f - f_h\|_1 \ \le\ C\omega^r(f;h)_1\,.$$

On the other hand, it follows from (1.12) and (2.8) that

$$\int_0^T t^{r/n-1}\widehat{f_h}^{\star}(t)\,\mathrm{d}t \ \le\ C\sum_{j=1}^{n}\big\|D_j^r(f_h)\big\|_1 \ \le\ Ch^{-r}\omega^r(f;h)_1\,.$$

This completes the proof. □

Inequality (2.14) is an exact counterpart of (2.10), but it arises from a completely different reasoning. We remark once more that (2.14) does not hold for $n = 1$.

If

$$\omega^r(f;\delta)_1 \ =\ O(\delta^r)\,, \tag{2.15}$$

then it follows from (2.14) that

$$\int_0^{\infty} t^{r/n-1}\hat{f}^{\star}(t)\,\mathrm{d}t \ \le\ \sup_{\delta>0}\delta^{-r}\omega^r(f;\delta)_1\,.$$

Note that if $f \in \widetilde{W}_1^r(\mathbb{R}^n)$, then (2.15) holds (see (2.3)), but the converse is not true.

By using Propositions 2.1 and 2.4, we obtain a weighted rearrangement inequality for the Fourier transform.

Theorem 2.5. *Let* $r \in \mathbb{N}$ *and* $1 < p \le 2,\ n \in \mathbb{N}$ *or* $p = 1,\ n \ge 2$. *Also, let* ρ *be a non–negative locally integrable function on* $[0, +\infty[$ *and set* $\lambda(t) = \int_0^t \rho(u)\,\mathrm{d}u$. *Suppose* *that* $\lambda(t)t^{1-p}$ *increases and* $\lambda(t)t^{-\alpha p}$ *decreases on* $]0,+\infty[$, *with* $\alpha = r/n + 1/p'$.[1] *Then, for each function* $f \in L^p(\mathbb{R}^n)$

$$\int_0^\infty \hat{f}^\star(t)^p \rho(t)\,\mathrm{d}t \ \le\ C \sum_{\nu\in\mathbb{Z}} \lambda(2^{n\nu})2^{n\nu(1-p)}\omega^r(f;2^{-\nu})_p^p \gamma_\nu\,, \tag{2.16}$$

where [2]

$$\gamma_\nu \ =\ \frac{1}{\omega_\nu}\min\left(\omega_\nu - \omega_{\nu+1}, \omega_\nu - 2^{-r}\omega_{\nu-1}\right), \tag{2.17}$$

with

$$\omega_\nu \ =\ \omega^r\big(f;2^{-\nu}\big)_p \quad (\nu\in\mathbb{Z})\,.$$

Proof. We follow the same scheme used in [Ko1]. We have

$$\begin{aligned}\int_0^\infty \hat{f}^\star(t)^p\rho(t)\,\mathrm{d}t \ &\le\ \sum_{\nu\in\mathbb{Z}} \hat{f}^\star(2^{n\nu})^p\big(\lambda\big(2^{n(\nu+1)}\big) - \lambda(2^{n\nu})\big)\\ &=\ \sum_{\nu\in\mathbb{Z}} \lambda(2^{n\nu})\Big(\hat{f}^\star\big(2^{n(\nu-1)}\big)^p - \hat{f}^\star(2^{n\nu})^p\Big)\\ &=\ \sum_{\nu\in\mathbb{Z}} d_\nu\varepsilon_\nu\,,\end{aligned}$$

where we have denoted $d_\nu = \lambda(2^{n\nu})2^{n\nu(1-p)}$ and

$$\varepsilon_\nu \ =\ 2^{n\nu(p-1)}\Big(\hat{f}^\star\big(2^{n(\nu-1)}\big)^p - \hat{f}^\star(2^{n\nu})^p\Big)\,.$$

Recall that, by our assumption, the sequence $\{d_\nu\}_{\nu\in\mathbb{Z}}$ is increasing. Also, by Propositions 2.1 and 2.4, we have

$$\sum_{j=\nu}^{\infty}\varepsilon_j \ \le\ C\omega_\nu^p\,, \qquad \sum_{j=-\infty}^{\nu} 2^{jrp}\varepsilon_j \ \le\ C2^{\nu rp}\omega_\nu^p\,. \tag{2.18}$$

Let

$$\sigma' \ =\ \big\{k : \omega_{k+1} \ge 2^{-r}\omega_{k-1}\big\}\,, \quad \sigma'' \ =\ \big\{k : \omega_{k+1} < 2^{-r}\omega_{k-1}\big\}\,.$$

Since the sequence $\{\omega_k\}_{k\in\mathbb{Z}}$ is bounded, it is clear that there exists some $k_0\in\mathbb{Z}$ such that $k\in\sigma'$ for all $k\le k_0$.

If $\mu\in\sigma'$ but $\mu+1\in\sigma''$, then

$$\gamma_\mu \ =\ (\omega_\mu - \omega_{\mu+1})/\omega_\mu\,, \quad \gamma_{\mu+1} \ =\ \big(\omega_{\mu+1} - 2^{-r}\omega_\mu\big)/\omega_{\mu+1}\,.$$

Thus,

$$\gamma_\mu + \gamma_{\mu+1} \ \ge\ 1 - 2^{-r} \quad \big(\mu\in\sigma',\ \mu+1\in\sigma''\big)\,. \tag{2.19}$$

Let ν and μ be integers such that $\nu<\mu$ and $[\nu,\mu]\cap\sigma'' = \emptyset$. We estimate the sum

$$S'_{\nu,\mu} \ =\ \sum_{k=\nu}^{\mu} d_k\varepsilon_k\,.$$

---

[1] Note that in view of the inequalities (1.5), (1.8) and (1.12), these conditions are quite natural.

[2] In view of (2.2) we have $\gamma_k \ge 0$. Of course, we assume that f is not equivalent to 0.

Let $a_k = d_k - d_{k-1}$ for $\nu + 1 \le k \le \mu$ and $a_\nu = d_\nu$. By (2.18)

$$\begin{aligned} S'_{\nu,\mu} &= \sum_{k=\nu}^{\mu} \varepsilon_k \sum_{j=\nu}^{k} a_j = \sum_{j=\nu}^{\mu} a_j \sum_{k=j}^{\mu} \varepsilon_k \le C \sum_{j=\nu}^{\mu} a_j \omega_j^p \\ &= C\left(\sum_{j=\nu}^{\mu} a_j \sum_{k=j}^{\mu} (\omega_k^p - \omega_{k+1}^p) + d_\mu \omega_{\mu+1}^p \right) \\ &\le C_p \left(\sum_{k=\nu}^{\mu} d_k \omega_k^p \gamma_k + d_\mu \omega_{\mu+1}^p \right). \end{aligned}$$

If $\mu + 1 \in \sigma''$, then by (2.19) we obtain

$$S'_{\nu,\mu} \le C \sum_{k=\nu}^{\mu+1} d_k \omega_k^p \gamma_k \,.$$

If all the integers $k > \nu$ belong to σ', then the convergence of the series in the right–hand side of (2.16) will imply that $\omega_\mu^p d_\mu \to 0$ as $\mu \to \infty$. Thus we get that

$$\sum_{k\in\sigma'} d_k \varepsilon_k \le C \sum_{k\in\mathbb{Z}} d_k \omega_k^p \gamma_k \,.$$

Now suppose that μ and ν are integers such that $\mu + 1 \le \nu$, $[\mu+1,\nu] \cap \sigma' = \emptyset$ and $\mu \in \sigma'$. Consider the sum

$$S''_{\mu,\nu} = \sum_{k=\mu+1}^{\nu} d_k \varepsilon_k \,.$$

Denote $\eta_k = 2^{kr}\omega_k$. Also, let $b_k = 2^{-krp} d_k - 2^{-(k+1)rp} d_{k+1}$ for $\mu + 1 \le k < \nu$ and $b_\nu = 2^{-\nu rp} d_\nu$ (recall that $2^{-krp} d_k$ is decreasing). Then by (2.18) and (2.19)

$$\begin{aligned} S''_{\mu,\nu} &= \sum_{k=\mu+1}^{\nu} 2^{krp} \varepsilon_k \sum_{j=k}^{\nu} b_j = \sum_{j=\mu+1}^{\nu} b_j \sum_{k=\mu+1}^{j} 2^{krp} \varepsilon_k \\ &\le C \sum_{j=\mu+1}^{\nu} b_j \eta_j^p \\ &= C \left(\sum_{j=\mu+1}^{\nu} b_j \sum_{k=\mu+1}^{j} \left(\eta_k^p - \eta_{k-1}^p\right) + 2^{-(\mu+1)rp} d_{\mu+1} \eta_\mu^p \right) \\ &\le C \sum_{k=\mu}^{\nu} d_k \omega_k^p \gamma_k \,. \end{aligned}$$

Thus we get

$$\sum_{k\in\sigma''} d_k \varepsilon_k \le C \sum_{k\in\mathbb{Z}} d_k \omega_k^p \gamma_k$$

and the proof is complete. □

Remark 2.6. From the estimates (2.12) and (2.13) it follows at once that

$$\int_0^\infty \hat{f}^\star(t)^p \rho(t)\,\mathrm{d}t \;\le\; C\int_0^\infty \rho(t)t^{1-p}\omega^r\big(f;t^{-1/n}\big)_p^p\,\mathrm{d}t\,. \tag{2.20}$$

The integral in the right–hand side of (2.20) is equivalent to the sum

$$\sum_{\nu\in\mathbb{Z}} \lambda(2^{n\nu})2^{n\nu(1-p)}\big(\omega_\nu^p-\omega_{\nu+1}^p\big)\,, \quad \omega_\nu \;=\; \omega^r(f;2^{-\nu})_p\,.$$

The estimate (2.20) is not sharp if $\omega^r(f;\delta)_p$ decreases very rapidly as $\delta\to 0$ (so that its order is close to the best possible $O(\delta^r)$). For example, if for some $\beta>0$

$$\omega^r(f;\delta)_p \;=\; O\big(\delta^r\log^\beta(1/\delta)\big)\,, \quad \delta\;\to\;0\,,$$

using (2.20) we must require that

$$\int_2^\infty \rho(t)t^{-\alpha p}(\log t)^{\beta p}\,\mathrm{d}t \;<\;\infty \qquad \left(\alpha=\frac{r}{n}+\frac{1}{p'}\right)$$

to make sure that the integral

$$\int_0^\infty \hat{f}^\star(t)^p\rho(t)\,\mathrm{d}t \tag{2.21}$$

is convergent. But if we use (2.16), we easily see that the weaker condition

$$\int_2^\infty \rho(t)t^{-\alpha p}(\log t)^{\beta p-1}\,\mathrm{d}t \;<\;\infty$$

already implies the convergence of (2.21)

Now we are going to prove that Theorem 2.5 is sharp.

Let $r\in\mathbb{N}$. We shall say that a function $\omega(\delta)$ defined on $[0,+\infty[$ belongs to the class Ω_r if it satisfies the following three conditions:

1. $\omega(\delta)\ge 0$ for all δ and $\omega(0)=0$;
2. $\omega(\delta)$ is increasing, continuous and bounded on $[0,+\infty[$;
3. $\omega(2\delta)\le 2^r\omega(\delta),\ 0\le\delta<\infty$.

Note that for each function $f\in L^p(\mathbb{R}^n)$ $(1\le p<\infty)$ its modulus of continuity $\omega^r(f;\delta)_p$ belongs to Ω_r (see (2.2)).

If $\omega\in\Omega_r$ and $1\le p<\infty$, then we shall denote by $L_p^{\omega,r}(\mathbb{R}^n)$ the class of all functions $f\in L^p(\mathbb{R}^n)$ such that

$$\omega^r(f;\delta)_p \;=\; O(\omega(\delta))\,.$$

Theorem 2.7. *Let $n,r\in\mathbb{N}$, $\omega\in\Omega_r$ and $1\le p\le 2$. Also, let $\rho(t)$ be a non–negative locally integrable function on $[0,\infty[$ and set $\lambda(t)=\int_0^t\rho(u)\,\mathrm{d}u$. Suppose that $\lambda(t)t^{1-p}$ increases and $\lambda(t)t^{-\alpha p}$ decreases $(\alpha=r/n+1/p')$. Let $\omega_\nu=\omega(2^{-\nu})$,*

$$\gamma_\nu \;=\; \frac{1}{\omega_\nu}\min\big(\omega_\nu-\omega_{\nu+1},\omega_\nu-2^{-r}\omega_{\nu-1}\big)\,. \tag{2.22}$$

If the series

$$\sum_{\nu=0}^{\infty} \lambda(2^{n\nu}) 2^{n\nu(1-p)} \omega_\nu^p \gamma_\nu \tag{2.23}$$

diverges, then there exists a function $f \in L_p^{\omega,r}(\mathbb{R}^n)$ *such that*

$$\int_1^\infty \hat{f}^\star(t)^p \rho(t)\,\mathrm{d}t \;=\; \infty\,. \tag{2.24}$$

Proof. Set $\eta_\nu = 2^{\nu r}\omega_\nu$. The sequence $\{\eta_\nu\}$ is increasing. It follows from the divergence of the series (2.23) that $\eta_\nu \to \infty$ as $\nu \to \infty$. Set $\nu_1 = 0$ and [3)]

$$\nu_{k+1} \;=\; \min\left\{\nu \,:\, \omega_\nu \le \frac{1}{2}\,\omega_{\nu_k} \;\text{ and }\; \eta_\nu \ge 2^r \eta_{\nu_k}\right\}. \tag{2.25}$$

For each $k \in \mathbb{N}$ at least one of the two following inequalities holds

$$\omega_{\nu_{k+1}} \;\ge\; \frac{1}{4}\,\omega_{\nu_k} \tag{2.26}$$

or

$$\eta_{\nu_{k+1}} \;\le\; 4^r \eta_{\nu_k}\,. \tag{2.27}$$

Furthermore, let

$$V_N(x) \;=\; \frac{1}{N^n} \prod_{j=1}^n \int_N^{2N} \frac{\sin 2\pi\lambda x_j}{\pi x_j}\,\mathrm{d}\lambda$$

be the De la Vallée–Poussin kernel. Then (see [N])

$$\|V_N\|_p \;\le\; C N^{1-1/p}\,. \tag{2.28}$$

Denote $U_k(x) = V_{2^{\nu_k}}(x)$ and consider the function

$$f(x) \;=\; \sum_{k=0}^{\infty} \omega_{\nu_k} 2^{n\nu_k(1/p-1)} U_k(x)\,. \tag{2.29}$$

It follows from (2.28) that the series in (2.29) converges in $L^p(\mathbb{R}^n)$. Let us estimate the L^p–modulus of continuity of f. Let $2^{-s-1} < h \le 2^{-s}$ ($s \in \mathbb{N}$). Using (2.3), (2.28) and the properties of the kernel V_n (see [N]), we have ($1 \le j \le n$)

$$\begin{aligned}
\left\|\Delta_j^r(h) f\right\|_p \;&\le\; \sum_{k=0}^{s} \omega_{\nu_k} 2^{n\nu_k(1/p-1)} \left\|\Delta_j^r(h) U_k\right\|_p + 2^r \sum_{k=s+1}^{\infty} \omega_{\nu_k} 2^{n\nu_k(1/p-1)} \|U_k\|_p \\
&\le\; C\left(h^r \sum_{k=0}^{s} \eta_{\nu_k} + \sum_{k=s+1}^{\infty} \omega_{\nu_k}\right) \\
&\le\; C\big(h^r \eta_{\nu_s} + \omega_{\nu_{s+1}}\big)\,.
\end{aligned}$$

Since the function $\delta^{-r}\omega(\delta)$ is almost decreasing, we get

$$\left\|\Delta_j^r(h) f\right\|_p \;\le\; C\omega(h)\,, \quad j \;=\; 1, \ldots, n\,.$$

3) Such sequences were used before by K. I. Oskolkov [O].

Thus, $f \in L_p^{\omega,r}(\mathbb{R}^n)$.

Furthermore, for every $N \in \mathbb{N}$, we have (see [N, p. 305]) $\widehat{V_N}(\xi) \geq 0$ ($\xi \in \mathbb{R}^n$) and $\widehat{V_N}(\xi) = 1$ if $|\xi_j| \leq N$, ($j = 1, \dots, n$). It follows that

$$\hat{f}^\star\big(2^{n(\nu_k+1)}\big) \;\geq\; \omega_{\nu_k} 2^{n\nu_k(1/p-1)}\,. \tag{2.30}$$

Using this estimate and denoting $\lambda_\nu = \lambda(2^{n\nu})$, we obtain

$$\begin{aligned}\int_1^\infty \hat{f}^\star(t)^p \rho(t)\,\mathrm{d}t \;&\geq\; \sum_{k=0}^\infty \hat{f}^\star(2^{n\nu_{k+1}})^p\big(\lambda_{\nu_{k+1}} - \lambda_{\nu_k}\big)\\ &\geq\; \sum_{k=1}^\infty \omega_{\nu_k}^p 2^{n\nu_k(1-p)}\big(\lambda_{\nu_k} - \lambda_{\nu_{k-1}}\big)\\ &\geq\; (1-2^{-p})\sum_{k=1}^\infty \omega_{\nu_k}^p 2^{n\nu_k(1-p)}\lambda_{\nu_k} - \omega^p(1)\lambda(1)\,.\end{aligned}$$

It remains to prove that

$$\sum_{k=1}^\infty \omega_{\nu_k}^p 2^{n\nu_k(1-p)}\lambda_{\nu_k} \;=\; \infty\,. \tag{2.31}$$

Denote $d_\nu = 2^{n\nu(1-p)}\lambda_\nu$. By our assumption, the sequence $\{d_\nu\}$ is increasing. Let

$$S_k \;=\; \sum_{\nu=\nu_k}^{\nu_{k+1}-1} d_\nu \omega_\nu^p \gamma_\nu\,.$$

Suppose that for some $k \in \mathbb{N}$, (2.26) holds. Then we have, using also (2.25)

$$S_k \;\leq\; d_{\nu_{k+1}} \sum_{\nu=\nu_k}^{\nu_{k+1}-1} \omega_\nu^{p-1}(\omega_\nu - \omega_{\nu+1}) \;\leq\; d_{\nu_{k+1}}\omega_{\nu_k}^p \;\leq\; 4^p d_{\nu_{k+1}}\omega_{\nu_{k+1}}^p\,.$$

Now suppose that (2.27) holds. Recall that the sequence $\{2^{-\nu rp}d_\nu\}$ is decreasing. Thus we get

$$S_k \;\leq\; \sum_{\nu=\nu_k}^{\nu_{k+1}-1} 2^{-\nu rp} d_\nu \eta_\nu^{p-1}(\eta_\nu - \eta_{\nu-1}) \;\leq\; 2^{-\nu_k rp} d_{\nu_k}\eta_{\nu_{k+1}}^p \leq 4^{rp} d_{\nu_k}\omega_{\nu_k}^p\,.$$

From these estimates and the divergence of the series (2.23), we get (2.31). The proof is now complete. □

Inequality (2.30) shows that for every p, such that $1 \leq p \leq 2$, every $n \in \mathbb{N}$ and every order for the modulus of continuity, the estimate (2.12) is sharp for the values $t = 2^{n\nu_k}$, where $\{\nu_k\}$ is the sequence defined by (2.25). At the same time, for $1 < p \leq 2$, $n \geq 1$ or $p = 1$, $n \geq 2$, this estimate can be strengthened in a certain "integral" sense (see (2.9), (2.10) and (2.14)). Now we shall show that for $p = n = 1$, the situation is quite different. In this case the estimate (2.13) is sharp for all values $t > 0$ simultaneously. For simplicity we shall consider the case $r = 1$.

Proposition 2.8. *For each $\omega \in \Omega_1$, there exists a function $f \in L_1^{\omega,1}(\mathbb{R})$, such that, for every $t > 0$*

$$\hat{f}^\star(t) \;\geq\; C\omega\left(\frac{1}{t}\right), \tag{2.32}$$

where C is a positive constant.

Proof. Denote $\omega_\nu = \omega(2^{-\nu})$, $\eta_\nu = 2^\nu \omega_\nu$. Let $\{\nu_k\}$ be the sequence defined by (2.25). Furthermore, let $\Delta_k = [0, 2^{-\nu_k}]$ and $f_k(x) = 2^{\nu_k}\omega_{\nu_k}\chi_{\Delta_k}(x)$. Let

$$f(x) \;=\; \sum_{k=0}^{\infty} f_k(x)\,,$$

where the series converges in $L^1(\mathbb{R})$.

Suppose $2^{-\nu_{s+1}} < h \leq 2^{-\nu_s}$. Then (see (2.25))

$$\begin{aligned}
\|\Delta(h)f\|_1 \;&\leq\; \sum_{k=0}^{s} \|\Delta(h)f_k\|_1 + 2\sum_{k=s+1}^{\infty} \|f_k\|_1 \\
&\leq\; 2\left(h\sum_{k=0}^{s} 2^{\nu_k}\omega_{\nu_k} + \sum_{k=s+1}^{\infty} \omega_{\nu_k}\right) \\
&\leq\; 8\omega(h)\,.
\end{aligned}$$

Thus $f \in L_1^{\omega,1}(\mathbb{R})$.

Next we have

$$\hat{f}(\xi) \;=\; \sum_{k=0}^{\infty} 2^{\nu_k}\omega_{\nu_k}\frac{1 - e^{-2\pi\imath\xi 2^{-\nu_k}}}{2\pi\imath\xi}\,.$$

Thus

$$\left|\hat{f}(\xi)\right| \;\geq\; \frac{1}{\pi\,|\xi|}\sum_{k=0}^{\infty} 2^{\nu_k}\omega_{\nu_k}\sin^2(\pi\xi 2^{-\nu_k})\,.$$

Let $2^{\nu_s} < t \leq 2^{\nu_{s+1}}$. For each $k = 0, 1, \ldots$, there holds at least one of the inequalities

$$(a)\quad 4\omega_{\nu_{k+1}} \;\geq\; \omega_{\nu_k} \qquad \text{or} \qquad (b)\quad \eta_{\nu_{k+1}} \;\leq\; 4\eta_{\nu_k}\,. \tag{2.33}$$

Suppose that, for $k = s$, we have (a). Then, for any $\xi \in [2^{\nu_{s+1}-2}, 2^{\nu_{s+1}-1}]$

$$\left|\hat{f}(\xi)\right| \;\geq\; \frac{1}{2\pi}\,\omega_{\nu_{s+1}} \;\geq\; \frac{1}{8\pi}\,\omega\left(\frac{1}{t}\right).$$

Thus

$$\hat{f}^\star\left(\frac{t}{4}\right) \;\geq\; \frac{1}{8\pi}\,\omega\left(\frac{1}{t}\right).$$

Now suppose that, for $k = s$, there holds the inequality (b) in (2.33). Let

$$E_t \;=\; \left\{\xi \in [t, 4t] \,:\, \left|\sin\left(\pi\xi 2^{-\nu_s}\right)\right| \geq \frac{1}{\sqrt{2}}\right\}.$$

It is easy to see that $|E_t| \geq t/4$. For each $\xi \in E_t$

$$\left|\hat{f}(\xi)\right| \;\geq\; \frac{1}{\pi\sqrt{2}\,\xi}\,2^{\nu_s}\omega_{\nu_s} \;\geq\; \frac{1}{16\pi\sqrt{2t}}\,2^{\nu_s+1}\omega_{\nu_{s+1}} \;\geq\; C\omega\left(\frac{1}{t}\right)$$

($C > 0$). Thus we get $\hat{f}^\star\left(\frac{t}{4}\right) \geq C\omega\left(\frac{1}{t}\right)$. And the proof is finished. □

3. Fourier transforms of functions or distributions in the Hardy spaces $H^p(\mathbb{R}^n)$

The notation $H^p(\mathbb{R}^n)$ will have for us the same meaning as in Chapter III, Section 4 of [G-R]. That is, $H^p(\mathbb{R}^n)$ will be a space of real–valued tempered distributions $f(x)$ in $\mathbb{R}^n$, which are the boundary values of harmonic functions $u(x,t)$ in the upper half space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \,]0,\infty[$, having a non–tangential maximal function $m_u(x) = \sup_{|y-x|<t} |u(y,t)|$ in $L^p(\mathbb{R}^n)$.

We shall use two other ways to look at H^p. First of all, in the one–dimensional case, each $f \in H^p(\mathbb{R})$ corresponds to a holomorphic function in $\mathbb{R}^2_+$, $F = u + \imath v$ having $\sup_{t>0}\left(\int_{\mathbb{R}} |F(x+\imath t)|^p \,\mathrm{d}x\right)^{1/p} < \infty$. This supremum is equivalent to $\|m_u\|_p$, which is, by definition, $\|f\|_{H^p}$. Still in the one–dimensional case, there is another quantity equivalent to $\|f\|_{H^p}$, which we shall find useful; namely $\left(\int_{\mathbb{R}} |F(x)|^p \,\mathrm{d}x\right)^{1/p}$, where $F(x) = \lim_{t\to 0} F(x+\imath t)$, a limit that is shown to exist for a.e. $x \in \mathbb{R}$. For this view of H^p see Chapter 1 of [G-R].

Finally, the other convenient way to look at H^p will be the atomic decomposition, that we shall need below. For this and additional information on Hardy spaces, we refer the reader to [G-R].

Let $f \in H^p(\mathbb{R}^n)$, $0 < p < \infty$. For $r \in \mathbb{N}$ and $1 \leq j \leq n$, we denote (see (2.1)) for $0 \leq \delta < \infty$,

$$\omega_j^r(f;\delta)_{H^p} \;=\; \sup_{0\leq h\leq\delta} \left\|\Delta_j^r(h)f\right\|_{H^p} , \quad \omega^r(f;\delta)_{H^p} \;=\; \sum_{j=1}^{n} \omega_j^r(f;\delta)_{H^p} .$$

If $f \in H^p(\mathbb{R}^n)$, $0 < p < \infty$, and there exists a derivative $D_j^r f \in H^p(\mathbb{R}^n)$ (in the sense of distributions), then

$$\omega_j^r(f;\delta)_{H^p} \;\leq\; C\delta^r \left\|D_j^r f\right\|_{H^p} . \tag{3.1}$$

This inequality was obtained in [Os] by using atomic decompositions.

We shall use the following refinement of the inequality (1.15).

Lemma 3.1. *Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $\varepsilon > 0$. Then there exists a constant C such that for every $f \in H^p(\mathbb{R}^n)$*

$$\left(\int_0^\infty t^{\varepsilon p-1} F_\varepsilon^\star(t)^p \,\mathrm{d}t\right)^{1/p} \;\leq\; C\,\|f\|_{H^p} , \tag{3.2}$$

where $F_\varepsilon(\xi) = |\xi|^{n(1-1/p-\varepsilon)}\left|\hat{f}(\xi)\right|$.

Proof. Set $\lambda = n(\varepsilon + 1/p - 1)$, $N = [n(1/p - 1)]$. It is sufficient to prove (3.2) for small ε, so we may assume that $N + 1 - \lambda \geq 0$.

First we consider the case when f is a unit p–atom (see [G-R]). In this case, it follows from (1.14) and the inequality $\left|\hat{f}(\xi)\right| \leq C\,|\xi|^{N+1}$ (see [T-W]), that

$$\|F_\varepsilon\|_\infty \;\leq\; C\,. \tag{3.3}$$

Furthermore, denote $E_t = \{\xi \in \mathbb{R}^n \,:\, F_\varepsilon(\xi) \geq F_\varepsilon^\star(t)\}$; then $|E_t| \geq t$. Let S_t be the complement of the ball in $\mathbb{R}^n$ of measure $t/2$ and centered at the origin. Then $|E_t \cap S_t| \geq t/2$ and we have

$$\begin{aligned} F_\varepsilon^\star(t)^p \;&\leq\; \frac{2}{t}\int_{E_t\cap S_t} |F_\varepsilon(\xi)|^p\,\mathrm{d}\xi \\ &\leq\; \frac{2}{t}\left(\int_{S_t} |\xi|^{-2\lambda p/(2-p)}\,\mathrm{d}\xi\right)^{1-p/2}\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{p/2} \\ &\leq\; Ct^{p(1/2-\varepsilon)-1}\,. \end{aligned}$$

Using this estimate and (3.3), we get

$$\left(\int_0^\infty t^{\varepsilon p-2}\,\mathrm{d}t\int_0^t F_\varepsilon^\star(u)^p\,\mathrm{d}u\right)^{1/p} \;\leq\; C\,. \tag{3.4}$$

After re–scaling, it follows that (3.4) holds with the same constant for any p–atom f.

In the general case, we shall use the atomic decomposition $f = \sum_j \lambda_j a_j$ (see [G-R]). We have $\hat{f} = \sum_j \lambda_j \widehat{a_j}$ and $F_\varepsilon(\xi) \leq \sum_j |\lambda_j| A_j(\xi)$, where $A_j(\xi) = |\xi|^{-\lambda}\,|\widehat{a_j}(\xi)|$. Moreover

$$\int_0^t F_\varepsilon^\star(u)^p\;\mathrm{d}u \;\leq\; \sum_j |\lambda_j|^p \int_0^t A_j^\star(u)^p\,\mathrm{d}u\,.$$

Thus the validity of (3.4) for p–atoms implies (3.2) for each $f \in H^p$. The lemma is proved. □

Remark 3.2. For the function $\varphi_0(\xi) = 1/|\xi|$ $(\xi \in \mathbb{R}^n)$ we have $\varphi_0^\star(t) = (v_n/t)^{1/n}$, where v_n is the measure of the n–dimensional unit ball. Thus it follows from (1.3) that (3.2) gives a strengthening of Hardy's inequality (1.15).

Remark 3.3. Note that $F_\varepsilon(\xi) = \left|\widehat{I_\lambda f}(\xi)\right|$, where $I_\lambda f$ is a Riesz potential of f. Thus, the inequality (3.2) can also be derived from the embedding theorems (see [T-W] and [F-R-S]) and the inequality (1.6). However, we chose to give a more direct proof.

Next we obtain

Corollary 3.4. *Let $f \in H^p(\mathbb{R}^n)$ $(0 < p \leq 1,\, n \in \mathbb{N})$ and $r \in \mathbb{N}$. Suppose that there exist partial derivatives $D_j^r f \in H^p(\mathbb{R}^n)$ $(j = 1, \ldots, n)$. Then*

$$\left(\int_0^\infty t^{rp/n-1} F^\star(t)^p\,\mathrm{d}t\right)^{1/p} \;\leq\; C\sum_{j=1}^n \left\|D_j^r f\right\|_{H^p}\,, \tag{3.5}$$

where $F(\xi) = |\xi|^{n(1-1/p)}\big|\hat{f}(\xi)\big|$.

Proof. Indeed, it is sufficient to apply Lemma 3.1 with $\varepsilon = r/n$ to each of the derivatives $D_j^r f$ and take into account (1.7). □

Inequality (3.5) is a particular case of the following theorem.

Theorem 3.5. *Let* $0 < p \le 1$ *and* $n, r \in \mathbb{N}$. *There exists a constant* C *such that for every* $f \in H^p(\mathbb{R}^n)$ *and every* $T > 0$

$$\left(\int_0^T t^{rp/n-1} F^\star(t)^p \,\mathrm{d}t\right)^{1/p} \le CT^{r/n}\omega^r\big(f; T^{-1/n}\big)_{H^p}, \tag{3.6}$$

where $F(\xi) = |\xi|^{n(1-1/p)}\big|\hat{f}(\xi)\big|$.

Proof. Set $\varphi_{\tau,j}(x) = \Delta_j^r(\tau) f(x)$ $(\tau > 0,\ j = 1, \ldots, n)$. Then

$$\widehat{\varphi}_{\tau,j}(\xi) = \left(e^{2\pi\imath\tau\xi_j} - 1\right)^r \hat{f}(\xi)\,. \tag{3.7}$$

Also, let $t \in]0, T]$ and $E_t = \{\xi \in \mathbb{R}^n : F(\xi) \ge F^\star(t)\}$. Then $|E_t| \ge t$. There exists $j \equiv j(t)$, such that the set $E'_t = \{\xi \in E_t \ :\ |\xi_j| \ge |\xi|/n\}$ has, at least, measure t/n. Denote $h = T^{-1/n}$. Suppose, first, that $E'_t \subset U$, where U is the ball of radius $T^{1/n}/2$ centered at the origin. Then, for every $\xi \in E'_t$, we have

$$\left|e^{2\pi\imath\xi_j h} - 1\right| \ \ge\ \left|\sin\left(\pi\xi_j h\right)\right| \ \ge\ 2\left|\xi_j\right| h \ \ge\ |\xi|\, h/n\,.$$

Denoting $\Phi_{\tau,j}(\xi) = |\xi|^{(1-1/p-r/n)n}|\widehat{\varphi}_{\tau,j}(\xi)|$, we get

$$F(\xi) \ \le\ (n/h)^r \Phi_{h,j}(\xi)\,, \quad \xi \in E'_t\,.$$

Since $F(\xi) \ge F^\star(t)$ for every $\xi \in E'_t$ and $|E'_t| \ge t/n$, it follows that

$$F^\star(t) \ \le\ (n/h)^r \Phi^\star_{h,j}(t/n)\,. \tag{3.8}$$

Now suppose that $E''_t \equiv E'_t \setminus U \ne \emptyset$. Since $|\xi_j| \ge (2nh)^{-1}$ for every $\xi \in E''_t$, then, as in the proof of Proposition 2.1, we have that

$$\int_0^{nrh} \left|e^{2\pi\imath\tau\xi_j} - 1\right|^r \mathrm{d}\tau \ \ge\ \frac{nrh}{2}\,, \quad \xi \in E''_t\,.$$

Thus (see (3.7)) for every $\xi \in E''_t$

$$F(\xi) \ \le\ \frac{2}{nrh}\int_0^{nrh} g_{\tau,j}(\xi)\,\mathrm{d}\tau\,,$$

where $g_{\tau,j}(\xi) = |\xi|^{n(1-1/p)}\,|\widehat{\varphi}_{\tau,j}(\xi)|$. By (1.14)

$$g_{\tau,j}(\xi) \ \le\ C\,\|\widehat{\varphi}_{\tau,j}\|_{H^p} \ \le\ C\omega^r(f;\tau)_{H^p}$$

and we obtain the inequality

$$F^\star(t) \ \le\ C\omega^r(f;h)_{H^p}\,. \tag{3.9}$$

Thus, we have proved that, in any case, there holds at least, one of the inequalities (3.8) or (3.9). Therefore, by applying Lemma 3.1, we get

$$\begin{aligned}\int_0^T t^{rp/n-1} F^\star(t)^p \,\mathrm{d}t \;&\le\; C\left[h^{-rp}\sum_{j=1}^n \int_0^T t^{rp/n-1}\Phi^\star_{h,j}(t)^p\,\mathrm{d}t + h^{-rp}\omega^r(f;h)^p_{H^p}\right]\\ &\le\; C' h^{-rp}\omega^r(f;h)^p_{H^p}\,.\end{aligned}$$

This completes the proof. □

Inequality (3.6) can also be expressed in terms of the Riesz potentials:

$$\left(\int_0^T t^{rp/n-1}\left(\widehat{I_\lambda f}\right)^\star(t)^p\,\mathrm{d}t\right)^{1/p} \;\le\; CT^{r/n}\omega^r\big(f;T^{-1/n}\big)_{H^p}\,, \tag{3.10}$$

where $\lambda = n(1/p-1)$.

It follows from (3.6) that, for every non–zero $f \in H^p(\mathbb{R}^n)$ $(0 < p \le 1)$

$$\omega^r(f;\delta)_{H^p} \;\ge\; C_f\delta^r \quad (\delta \ge 0)\,, \tag{3.11}$$

where C_f is some positive constant depending on f.

The analogous property of the modulus of continuity in $L^p(\mathbb{R}^n)$ is well–known — it follows from (2.2). But for $0 < p < 1$ this property can fail for functions in L^p $\big($for example if $f(x) = \chi_{[0,1]}(x)$ $(x \in \mathbb{R})$, then $\omega(f;\delta) \le (2\delta)^{1/p}$ for every $0 < p < \infty\big)$.

In fact, a stronger inequality than (3.11) does hold. Namely, for every $p > 0$

$$\omega^r(f;k\delta)_{H^p} \;\le\; Ck^r\omega^r(f;\delta)_{H^p} \quad (k \in \mathbb{N})\,, \tag{3.12}$$

where the constant C depends on p, r and n only. For the spaces H^p in the unit disk (3.12) follows from the results of P. Oswald [Os]; by similar methods it can be obtained for the functions in $H^p(\mathbb{R}^n)$ (see [C] and [So]).

Suppose that ω is a function belonging to the class Ω_r, $r \in \mathbb{N}$. By $H^{\omega,r}_p(\mathbb{R}^n)$ $(0 < p \le 1)$ we shall denote the class of all $f \in H^p(\mathbb{R}^n)$ satisfying the condition

$$\omega^r(f;\delta)_{H^p} \;=\; O(\omega(\delta))\,.$$

Note that, for every $f \in H^p(\mathbb{R}^n)$, its modulus of continuity is equivalent to some $\omega \in \Omega_r$. Indeed, set

$$\overline{\omega}(f;\delta) \;=\; \delta^r \sup_{h\ge\delta} h^{-r}\omega^r(f;h)_{H^p}\,.$$

Then $\overline{\omega}(f;\delta)$ increases and $\overline{\omega}(f;\delta)\delta^{-r}$ decreases in $]0,+\infty[$; therefore, $\overline{\omega}(f;\delta)$ belongs to Ω_r. Furthermore, in view of (3.12)

$$\omega^r(f;\delta)_{H^p} \;\le\; \overline{\omega}(f;\delta) \;\le\; C\omega^r(f;\delta)_{H^p}\,.$$

If $\omega \in \Omega_r$, then we set $\omega_\nu = \omega(2^{-\nu})$ $(\nu \in \mathbb{Z})$ and define the sequence $\{\gamma_\nu\}_{\nu\in\mathbb{Z}}$ by (2.22).

Theorem 3.6. *Let $0 < p \leq 1$, and let n and r be positive integers and $\omega \in \Omega_r$. Suppose that $\rho(t)$ is a non–negative, locally integrable function on $[0,+\infty[$, such that, if we denote $\lambda(t) = \int_0^t \rho(u)\,\mathrm{d}u$, the function $t^{-rp/n}\lambda(t)$ is decreasing in $[0,+\infty[$. Then*

(1) *if*

$$D_{n,p,r}(\omega,\rho) \;\equiv\; \sum_{\nu\in\mathbb{Z}} \lambda(2^{n\nu})\omega_\nu^p\gamma_\nu \;<\; \infty\,, \tag{3.13}$$

then, for every $f \in H_p^{\omega,r}(\mathbb{R}^n)$

$$\int_0^\infty F^\star(t)^p\rho(t)\,\mathrm{d}t \;\leq\; CD_{n,p,r}(\omega,\rho)\,, \tag{3.14}$$

where $F(\xi) = |\xi|^{n(1-1/p)}\big|\hat{f}(\xi)\big|$;

(2) *if $D_{n,p,r}(\omega,\rho) = \infty$, then there exists $f \in H_p^{\omega,r}(\mathbb{R}^n)$, such that*

$$\int_1^\infty F^\star(t)^p\rho(t)\,\mathrm{d}t \;=\; \infty\,. \tag{3.15}$$

Proof. (1) We have

$$\int_0^\infty F^\star(t)^p\rho(t)\,\mathrm{d}t \;\leq\; \sum_{\nu\in\mathbb{Z}} \lambda_\nu\varepsilon_\nu\,,$$

where $\lambda_\nu = \lambda(2^{n\nu})$ and $\varepsilon_\nu = F^\star\big(2^{-n(\nu-1)}\big)^p - F^\star(2^{-n\nu})^p$ ($\nu \in \mathbb{Z}$). It follows from (3.6) that

$$\sum_{j=\nu}^{\infty} \varepsilon_j \;\leq\; C\omega_\nu^p\,, \qquad \sum_{j=-\infty}^{\nu} 2^{jrp}\varepsilon_j \;\leq\; C2^{\nu rp}\omega_\nu^p\,.$$

The argument continues as in Theorem 2.5.

(2) Denote by $\widetilde{H}^p(\mathbb{R}^n)$ the space of all complex–valued distributions on $\mathbb{R}^n$ such that their real and imaginary parts belong to $H^p(\mathbb{R}^n)$, and set

$$\|f\|_{\widetilde{H}^p(\mathbb{R}^n)} \;=\; \|\Re f\|_{H^p(\mathbb{R}^n)} + \|\Im f\|_{H^p(\mathbb{R}^n)}\,.$$

Choose an integer $m > 1/p$ and, for each $\sigma > 0$, consider the function

$$g_\sigma(x) \;=\; \sigma^{1-m}\left(\frac{\sin(2\pi\sigma x)}{\pi x}\right)^m e^{2\pi\sigma m\imath x}\,, \quad x \in \mathbb{R}\,.$$

This function and all its derivatives belong to $\widetilde{H}^p(\mathbb{R})$. Moreover, we can estimate their $\widetilde{H}^p$–norms by L^p–norms. That way we easily get that

$$\big\|g_\sigma^{(k)}\big\|_{\widetilde{H}^p(\mathbb{R})} \;\leq\; C\sigma^{k+1-1/p} \quad (k = 0,1,\ldots)\,. \tag{3.16}$$

Also set

$$T_\sigma(x) \;=\; \prod_{j=1}^{n} g_\sigma(x_j)\,, \quad x \in \mathbb{R}^n\,.$$

This function belongs to $\widetilde{H}^p(\mathbb{R}^n)$. Moreover, by (3.16), we have

$$\|T_\sigma\|_{\widetilde{H}^p(\mathbb{R}^n)} \;\le\; C\sigma^{n(1-1/p)} \tag{3.17}$$

and

$$\left\|D_j^r T_\sigma\right\|_{\widetilde{H}^p(\mathbb{R}^n)} \;\le\; C\sigma^{r+n(1-1/p)}\,, \quad j \;=\; 1\,,\ldots,n\,. \tag{3.18}$$

Let us estimate the Fourier transform $\widehat{T_\sigma}$ from below. Observe that

$$\frac{\sin(2\pi\sigma x)}{\pi x} \;=\; \widehat{\chi_{[-\sigma,\sigma]}}(x)\,, \quad x \in \mathbb{R}\,.$$

Set $Q(\sigma) = [-\sigma,\sigma]^n$, $\varphi_\sigma = \chi_{Q(\sigma)}$ and $\Phi_\sigma = \varphi_\sigma \star \cdots \star \varphi_\sigma$ (m–fold convolution). Then

$$T_\sigma(x) \;=\; \sigma^{(1-m)n}\widehat{\Phi_\sigma}(x)e^{2\pi\sigma m\imath(x_1+\cdots+x_n)}$$

and

$$\widehat{T_\sigma}(\xi) \;=\; \Phi_\sigma(\sigma m-\xi_1,\ldots,\sigma m-\xi_n)\sigma^{(1-m)n}\,.$$

It is easy to see that

$$\Phi_\sigma(\xi) \;\ge\; \left(\frac{\sigma}{2}\right)^{(m-1)n} \quad \text{for all} \quad \xi \in Q\bigl(2^{1-m}\sigma\bigr)\,.$$

Thus, we have that $\widehat{T_\sigma}(\xi) \ge 0$ for all $\xi \in \mathbb{R}^n$ and

$$\widehat{T_\sigma}(\sigma m-\xi_1,\ldots,\sigma m-\xi_n) \;\ge\; 2^{(1-m)n}\,, \quad \text{if} \quad \xi \in Q\bigl(2^{1-m}\sigma\bigr)\,. \tag{3.19}$$

Now let $\eta_\nu = 2^{\nu p}\omega_\nu$. The sequence $\{\eta_\nu\}_{\nu=0}^\infty$ is increasing. It follows from the divergence of the series (3.13) that $\eta_\nu \to \infty$ as $\nu \to \infty$. Let $\{\nu_k\}$ be the sequence of integers defined by (2.25). Set $\tau_k = T_{2^{\nu_k+m}}$ and

$$f_0(x) \;=\; \sum_{k=1}^\infty \omega_{\nu_k}2^{\nu_k n(1/p-1)}\tau_k(x)\,, \quad x \in \mathbb{R}^n\,.$$

In view of (3.17), this series converges in $\widetilde{H}^p(\mathbb{R}^n)$. As in the proof of Theorem 2.7, we get that the real and imaginary parts of f_0 belong to $H_p^{\omega,r}(\mathbb{R}^n)$.

Now, it follows from (3.19), that

$$\widehat{f}_0(\gamma_k-\xi_1,\ldots,\gamma_k-\xi_n) \;\ge\; C\omega_{\nu_k}2^{\nu_k n(1/p-1)} \quad (c>0)$$

for every $\xi \in Q\bigl(2^{\nu_k+1}\bigr)$, where $\gamma_k = m2^{\nu_k+m}$. Let $F_0(\xi) = \bigl|\widehat{f}_0(\xi)\bigr|\,|\xi|^{n(1-1/p)}$. Then

$$F_0(\gamma_k-\xi_1,\ldots,\gamma_k-\xi_n) \;\ge\; c'\omega_{\nu_k} \quad \text{for all} \quad \xi \in Q\bigl(2^{\nu_k+1}\bigr)\,.$$

From here it follows that $F_0^\star\bigl(2^{n(\nu_k+1)}\bigr) \ge c'\omega_{\nu_k}$ ($c'>0$). Using this estimate, we obtain

$$\begin{aligned}
\int_1^\infty F_0^\star(2t)^p\rho(t)\,\mathrm{d}t \;&\ge\; \sum_{k=1}^\infty(\lambda_{\nu_{k+1}}-\lambda_{\nu_k})F_0^\star\bigl(2^{n\nu_{k+1}+1}\bigr)^p \\
&\ge\; c'\sum_{k=1}^\infty\,(\lambda_{\nu_{k+1}}-\lambda_{\nu_k})\omega_{\nu_{k+1}}^p \\
&\ge\; c'(1-2^{-p})\left(\sum_{k=2}^\infty\lambda_{\nu_k}\omega_{\nu_k}^p-\lambda(1)\omega(1)^p\right)\,.
\end{aligned}$$

Observe now that

$$\sum_{k=1}^{\infty} \lambda_{\nu_k} \omega_{\nu_k}^p \; = \; \infty \, .$$

This follows from the divergence of the series (3.13) by the same argument that the one in the proof of Theorem 2.7. The proof is completed □

Thus, condition (3.13) is necessary and sufficient in order to have

$$\int_0^\infty F^\star(t)^p \rho(t) \, \mathrm{d}t \; < \; \infty$$

for every $f \in H_p^{\omega,r}(\mathbb{R}^n)$.

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References

[B] Bari, N. K.: A Treatise on Trigonometric Series Vol 1 – 2, Pergamon Press, New York, 1964

[B-S] Bennett, C., and Sharpley, R.: Interpolation of Operators, Academic Press, 1988

[B-L] Bergh, J., and Löfström, J.: Interpolation Spaces. An Introduction, Springer–Verlag, New York, 1976

[B-I-N] Besov, O. V., Il'in, V. P., and Nikol'skiĭ, S. M.: Integral Representation of Functions and Imbedding Theorems, Vol 1 – 2, Winston, Washington D. C., Halsted, New York – Toronto – London, 1978

[Bo1] Bourgain, J.: A Hardy Inequality in Sobolev Spaces, Vrije University, Brussels, 1981

[Bo2] Bourgain, J.: Some Examples of Multipliers in Sobolev Spaces, IHES, 1985

[C] Colzani, L.: Jackson Theorems in Hardy Spaces and Approximation by Riesz Means, J. Appr. Theory **49**, No. 3 (1987), 240 – 251

[F-R-S] Fefferman, C., Riviere, N. M., and Sahger, Y.: Interpolation between H^p – Spaces: The Real Method, Trans. Amer. Math. Soc. **191** (1974), 75 – 81

[F-S] Fefferman, C., and Stein, E. M.: H^p Spaces of Several Variables, Acta Math. **129** (1972), 137 – 192

[G-R] García–Cuerva, J., and Rubio de Francia, J. L.: Weighted Norm Inequallities and Realated Topics, North Holland Math. Studies **116**, Amsterdam, 1985

[H] Hunt, R. A.: On $L(p, q)$ Spaces, Enseign. Math. **12** (1966), 249 – 276

[J-S 1] Jurkat, W. B., and Sampson, G.: On Maximal Rearrangement Inequalities for the Fourier Transform, Trans. Amer. Math. Soc. **282** (1984), 625 – 643

[J-S 2] Jurkat, W. B., and Sampson, G.: On Rearrangement and Weighted Inequalities for the Fourier Transform, Indiana Univ. Math. J. **33,** No. 2 (1984), 257 – 270

[Ko1] Kolyada, V. I.: Estimates of Rearrangements and Embedding Theorems, Mat. Sb. **136,** No. 178 (1988), 3 – 23; English transl. in Math. USSR Sb. **64,** No. 1 (1989), 1 – 21

[Ko2] Kolyada, V. I.: Estimates of Fourier Transforms in Sobolev Spaces, Studia Math. **125,** No. 1 (1997), 67 – 74

[N] Nikol'skiĭ, S. M.: Approximation of Functions of Several Variables and Imbedding Theorems, Springer–Verlag, Berlin – Heidelberg – New York, 1975

[O] Oskolkov, K. I.: Approximation Properties of Summable Functions on Sets of Full Measure, Math. USSR Sbornik Vol. **32,** No. 4 (1977), 489 – 514

[Os] Oswald, P.: On Some Approximation Properties of Real Hardy Spaces ($0 < p \leq 1$), J. Appr. Theory **40** (1966), 249 – 276

[P-W] Pełczyński, A., and Wojciechowski, M.: Molecular Decompositions and Embedding Theorems for Vector–Valued Sobolev Spaces with Gradient Norm, Studia Math. **107** (1993), 61 – 100

[Pi] Pietsch, A.: Eigenvalues and s–Numbers, Cambridge University Press, Cambridge – London – New York, 1987

[So] Soljanik, A. A.: On the Order of Approximation to Functions of $H^p(\mathbb{R}^n)$ ($0 < p \leq 1$) by Certain Means of Fourier Integrals, Analysis Mathematica **12**, No. 1 (1986), 59 – 75

[St] Stein, E. M.: Interpolation of Linear Operators, Trans. Amer. Math. Soc. **83** (1956), 482 – 492

[Sz] Szasz, O.: Fourier Series and Mean Moduli of Continuity, Trans. Amer. Math. Soc. **42,** No. 3 (1937), 355 – 395

[T-W] Taibleson, M. H., and Weiss, G. L.: The Molecular Characterization of Certain Hardy Spaces, Astérisque **77** (1980)

[Z] Zygmund, A.: Trigonometric Series, Second Edition, Cambridge University Press, Cambridge – London – New York, 1968

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