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# PALEY TYPE INEQUALITIES FOR ORTHOGONAL SERIES WITH VECTOR-VALUED COEFFICIENTS

J. GARCIA-CUERVA, K. S. KAZARIAN (Madrid) and V. I. KOLYADA(La Rioja)*

**Abstract.** We investigate the extension to Banach-space-valued functions of the classical inequalities due to Paley for the Fourier coefficients with respect to a general orthonormal system $\Phi$. This leads us to introduce the notions of Paley $\Phi$-type and $\Phi$-cotype for a Banach space and some related concepts. We study the relations between these notions of type and cotype and those previously defined. We also analyze how the interpolation spaces inherit these characteristics from the original spaces, and use them to obtain sharp coefficient estimates for functions taking values in Lorentz spaces.

## §1. Introduction

The aim of the present paper is to study estimates for the Fourier coefficients of vector-valued functions. It is part of a field, well developed by now, which we can call vector-valued Fourier analysis. The origins of this field lie in the work of S. Bochner [5]. In that work, along with the integration theory for functions taking values in an arbitrary Banach space, some questions of Fourier analysis were considered. In particular, it was shown that Plancherel theorem may fail for vector-valued functions.

Another classical result in the theory of Fourier transforms, the Hausdorff–Young theorem, served as the starting point for the work of J. Peetre [18]. He considered Banach spaces $B$, satisfying the following property: for a given $p \in (1,2]$ and every function $f : \mathbf{R} \to B$ such that $f \in L^p_B$

$$\left( \int_{\mathbf{R}} \big\| \widehat{f}(\xi) \big\|_B^{p'} \, d\xi \right)^{1/p'} \leqq A_p \left( \int_{\mathbf{R}} \big\| f(x) \big\|_B^{p} \, dx \right)^{1/p},$$

where $p' = p/(p-1)$, the conjugate exponent for $p$. We shall use this notation systematically, that is, if $p \in (1,\infty)$, then we shall denote its conjugate exponent by $p'$. Such spaces were called spaces of Fourier type $p$. In the work [18] this notion was used in a study of interpolation methods. A detailed review of the results pertaining to this line of research and further historical comments can be found in the paper [9], which we shall often refer to.

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As in the classical situation, we shall consider both the problem of estimating the Fourier coefficients of a function from estimates for the function itself, and conversely, estimating the function from estimates for its Fourier coefficients. First of all we shall recall the fundamental results for scalar-valued functions.

Let $\Phi = \{\varphi_n\}$ be an orthonormal system on an interval $[a,b]$, uniformly bounded on this interval, that is, such that

$$\left|\varphi_n(x)\right| \leqq M \qquad \left(x \in [a,b],\ n \in \mathbf{N}\right)$$

for some constant $M$ independent of $n$ and $x$.

THEOREM A. *For every $p \in [1,2]$, we have:*

(1) *If $\{c_k\} \in l^p$, then there exists a function $f \in L^{p'}[a,b]$ such that $c_k$ are the Fourier coefficients of $f$ with respect to the system $\Phi$ and*

$$\|f\|_{L^{p'}} \leqq M^{2/p-1} \left\| \{c_k\} \right\|_{l^p}. \tag{1.1}$$

(2) *If $f \in L^p[a,b]$, and $c_k$ are the Fourier coefficients of $f$ with respect to the system $\Phi$, then*

$$\left\| \{c_k\} \right\|_{l^{p'}} \leqq M^{2/p-1} \|f\|_{L^p}. \tag{1.2}$$

This theorem was proved by F. Hausdorff and W. H. Young for the trigonometric system and by F. Riesz in the general case.

Inequalities (1.1) and (1.2) can be strengthened in a certain sense.

THEOREM B. *For $1 < p \leqq 2$, we have:*

(1) *If $\{c_k\} \in l^p$, then $\{c_k\}$ is the sequence of the Fourier coefficients with respect to the system $\Phi$, of a function $f \in L^1[a,b]$, such that*

$$\left( \int_0^{b-a} t^{p-2} \left( f^*(t) \right)^p (t)\, dt \right)^{1/p} \leqq A_p M^{2/p-1} \left( \sum_{k=1}^{\infty} |c_k|^p \right)^{1/p}. \tag{1.3}$$

(2) *If $f \in L^p[a,b]$ and $c_k$ are the Fourier coefficients of $f$ with respect to the system $\Phi$, then*

$$\left( \sum_{k=1}^{\infty} k^{p-2} (c_k^*)^p \right)^{1/p} \leqq A_p M^{2/p-1} \left( \int_a^b \left| f(x) \right|^p \right)^{1/p}. \tag{1.4}$$

Here $f^*$ is the non-increasing rearrangement of the function $f$ and $\{c_k^*\}$ is the non-increasing rearrangement of the sequence $\{c_k\}$.

We also have a theorem dual to Theorem B, namely

THEOREM C. *Let $2 \leqq q < \infty$. Then:*
(1) *If a sequence $\{c_k\}$ satisfies the condition*

$$\sum_{k=1}^{\infty} k^{q-2}(c_k^*)^q < \infty,$$

*then there exists a function $f \in L^q[a,b]$, such that $\{c_k\}$ is the sequence of its Fourier coefficients with respect to the system $\Phi$ and*

$$\left(\int_a^b \left|f(x)\right|^q dx\right)^{1/q} \leqq A'_q M^{1-2/q}\left(\sum_{k=1}^{\infty} k^{q-2}(c_k^*)^q\right)^{1/q} \tag{1.5}$$

(2) *If $f \in L^1[a,b]$ and $c_k$ are the Fourier coefficients of $f$ with respect to the system $\Phi$, then*

$$\left(\sum_{k=1}^{\infty} |c_k|^q\right)^{1/q} \leqq A'_q M^{1-2/q}\left(\int_0^{b-a} t^{q-2}\big(f^*(t)\big)^q dt\right)^{1/q}. \tag{1.6}$$

Theorems B and C are due to Paley; for the trigonometric system they were proved by Hardy and Littlewood. Each of these theorems implies Theorem A as a corollary, but with additional multipliers on the right hand sides of the inequalities (see [21, Ch. 12]).

It is natural to study similar relations in the vector-valued case. Theorem A leads to the notions of Riesz type $p$ and strong Riesz cotype $p'$ $(1 < p \leqq 2)$ of a Banach space $B$ with respect to the orthonormal system $\Phi$ (these are the properties of $B$ expressed in inequalities of the form (1.1) and (1.2) respectively, see [9]). In order to study the possibility of strengthening these inequalities in the sense of Hardy–Littlewood–Paley, we introduce in the present paper the notions of type and cotype based on Theorems B and C.

Briefly, the content of the paper is as follows. In §2 we give the definitions of Paley type and cotype and some simple results related to these notions. In §3 we study the notions of Paley type and cotype for the trigonometric system and the Walsh system. We prove that, for any of these particular systems, if a Banach space $B$ is of Paley type $p$, then it is also of Paley cotype $p'$ and vice versa. It is also shown that the Rademacher system has the best indices of Paley type and cotype among all uniformly bounded orthonormal systems. In §4 we show that the spaces $L^p, 1 < p \leqq 2$ are of Paley type $p$ and strong Paley cotype $p'$. This assertion is stronger than the known Hausdorff–Young type inequalities for $L^p$-valued functions. The dual statement is proved for the case $2 \leqq p < \infty$.

In §5 we study the spaces obtained by real interpolation from the spaces with a given Paley type (Paley cotype). Estimates for the Fourier coefficients of functions taking values in these spaces are obtained.

Using the results of §5, we derive in §6 sharp coefficient inequalities for $L^{p,r}$-valued functions, where $L^{p,r}$ is a Lorentz space. These inequalities coincide with the known Hausdorff–Young type estimates only in the case $r = p'$.

## §2. Types and cotypes of the Banach spaces

First we recall some general definitions (see [9] for details). Let $(I,\mu)$ be a fixed measure space; throughout this paper we will assume that the measure $\mu$ is non-atomic and $\mu(I) = 1$. Furthermore, let $B$ be a Banach space. If $f : I \to B$ is a strongly measurable function, then by $f^*(t)$ $\big(t \in (0,1)\big)$ we denote the non-increasing rearrangement of the function $\big\| f(x) \big\|$ $(x \in I)$. The notation $L^{p,r}_B \equiv L^{p,r}(B)$ $(0 < p, r < \infty)$ stands for the Lorentz space of all strongly measurable functions $f : I \to B$ such that

$$\|f\|_{L^{p,r}_B} \equiv \left( \int_0^1 \left[ t^{1/p} f^*(t) \right]^r \frac{dt}{t} \right)^{1/r} < \infty.$$

We have $L^{p,p}_B = L^p_B$ $(0 < p < \infty)$.

If the sequence $\{b_k\}$ $(b_k \in B)$ satisfies the condition

$$\lim_{k\to\infty} b_k = 0, \tag{2.1}$$

then $\{b^*_k\}$ will denote the non-increasing rearrangement of the sequence $\big\{ \|b_k\| \big\}$. The Lorentz space $l^{p,r}_B \equiv l^{p,r}(B)$ $(0 < p, r < \infty)$ is the space of all sequences $\{b_k\}$ $(b_k \in B)$ satisfying (2.1) and such that

$$\big\| \{b_k\} \big\|_{l^{p,r}_B} \equiv \left( \sum_{k=1}^\infty k^{r/p-1} (b^*_k)^r \right)^{1/r} < \infty.$$

Here also $l^{p,p}_B = l^p_B$ $(0 < p < \infty)$.

Next, let $\Phi = \{\varphi_n\}$ be an orthonormal system (ONS) of real or complex-valued functions on $I$. We shall assume that

$$\big|\varphi_n(x)\big| \leqq M \quad \text{for all } x \in I \text{ and all } n \in \mathbf{N}. \tag{2.2}$$

For each function $f \in L^1_B$ set

$$c_n(f) = \int_I f(x)\overline{\varphi_n(x)}\, d\mu, \quad n \in \mathbf{N}. \tag{2.3}$$

DEFINITION 2.1. Let $1 \leqq p \leqq 2$. We shall say that a Banach space $B$ is of Riesz $\Phi$-type $p$, if there exists a constant $A$ such that for every finite sequence $\{b_k\}_{k=1}^n$ of elements $b_k \in B$

$$\left( \int_I \left\| \sum_{k=1}^n b_k \varphi_k(x) \right\|^{p'} d\mu \right)^{1/p'} \leqq A \left( \sum_{k=1}^n \|b_k\|^p \right)^{1/p}. \tag{2.4}$$

REMARK 2.2. Suppose that $B$ is of Riesz $\Phi$-type $p$. Then it is easy to see that for each sequence $b = \{b_k\} \in l_B^p$ the series $\sum_{k=1}^\infty b_k \varphi_k$ converges in the norm of $L_B^{p'}$ and its sum $f$ satisfies the inequality

$$\|f\|_{L_B^{p'}} \leqq A \|b\|_{l_B^p}.$$

Moreover, every permutation of this series converges in $L_B^{p'}$ to the same sum $f$.

DEFINITION 2.3. Let $1 \leqq p \leqq 2$. A Banach space $B$ is said to be of:

(1) Riesz $\Phi$-cotype $p'$, if there exists a constant $A$ such that for every polynomial $\sum_{k=1}^n b_k \varphi_k$ $(b_k \in B)$

$$\left( \sum_{k=1}^n \|b_k\|^{p'} \right)^{1/p'} \leqq A \left( \int_I \left\| \sum_{k=1}^n b_k \varphi_k(x) \right\|^p d\mu \right)^{1/p}; \tag{2.5}$$

(2) strong Riesz $\Phi$-cotype $p'$, if there exists a constant $A$ such that for every function $f \in L_B^p$

$$\left( \sum_{k=1}^\infty \| c_n(f) \|^{p'} \right)^{1/p'} \leqq A \left( \int_I \| f(x) \|^p d\mu \right)^{1/p}, \tag{2.6}$$

where $c_n(f)$ are the Fourier coefficients of $f$.

REMARK 2.4. These definitions are directly connected with the Riesz inequalities (1.1) and (1.2). For the Rademacher system the notions of type and cotype were studied first by J. Hoffman-Jorgensen [12] and B. Maurey [17] (see also [19]). A few years earlier J. Peetre [18] introduced the definition of Fourier type of a Banach space (in terms of the vector-valued Fourier transform). For general ONS the notions of type and cotype were considered in [9], where additional references and comments can be found.

Now, motivated by Theorem B, we introduce the notions of Paley type and cotype, which express stronger properties of the Banach spaces.

Definition 2.5. Let $1 < p < 2$. We say that a Banach space $B$ is of Paley $\Phi$-type $p$, if there exists a constant $A$ such that for every finite sequence $\{b_k\}_{k=1}^n$ $(b_k \in B)$

$$\left( \int_0^1 t^{p-2} \big( f^*(t) \big)^p \, dt \right)^{1/p} \leqq A \left( \sum_{k=1}^n \|b_k\|^p \right)^{1/p}, \tag{2.7}$$

where $f = \sum_{k=1}^n b_k \varphi_k$.

Inequality (2.7) can also be written in the form

$$\|f\|_{L_B^{p',p}} \leqq A \|b\|_{l_B^p}, \quad b = \{b_k\}_{k=1}^n. \tag{2.8}$$

For the Paley type, an observation similar to Remark 2.2 also holds.

Definition 2.6. Let $1 < p < 2$. A Banach space $B$ is said to be of:

(1) Paley $\Phi$-cotype $p'$, if there exists a constant $A$ such that for every polynomial $f = \sum_{k=1}^n b_k \varphi_k$ $(b_k \in B)$

$$\left( \sum_{k=1}^n k^{p-2} (b_k^*)^p \right)^{1/p} \leqq A \left( \int_I \left\| \sum_{k=1}^n b_k \varphi_k \right\|^p d\mu \right)^{1/p}; \tag{2.9}$$

(2) strong Paley $\Phi$-cotype $p'$, if there exists a constant $A$ such that for every function $f \in L_B^p$

$$\left( \sum_{k=1}^\infty k^{p-2} \big( c_k^*(f) \big)^p \right)^{1/p} \leqq A \left( \int_I |f|^p \, d\mu \right)^{1/p}. \tag{2.10}$$

Of course, this definition can also be written in terms of Lorentz norms; for example, inequality (2.10) takes the form

$$\big\| c(f) \big\|_{l_B^{p',p}} \leqq A \|f\|_{L_B^p}, \quad c(f) = \big\{ c_k(f) \big\}. \tag{2.11}$$

Remark 2.7. For $1 < p < 2$, we have (see [2, p. 217])

$$\|f\|_{L_B^{p'}} \leqq C_p \|f\|_{L_B^{p',p}}.$$

Thus, if a Banach space $B$ is of Paley $\Phi$-type $p$, then it is also of Riesz $\Phi$-type $p$. Of course, the converse is not true (see §6 below).

Similarly, if the space $B$ is of Paley $\Phi$-cotype $p'$, then it is also of Riesz $\Phi$-cotype $p'$.

We shall say that $\Phi$ is a complete system in $L^{p,r}(0<p,r<\infty)$, if all the polynomials $\sum_{k=1}^{n} \alpha_k \varphi_k$ with scalar coefficients form a dense set in $L^{p,r}$.

PROPOSITION 2.8. *Let $1<p<2$, and suppose that $\Phi$ is a uniformly bounded system which is complete in $L^p(I)$. Then:*

(1) *if a Banach space $B$ is of Riesz $\Phi$-cotype $p'$, then $B$ is of strong Riesz $\Phi$-cotype $p'$;*

(2) *if a Banach space $B$ is of Paley $\Phi$-cotype $p'$, then $B$ is of strong Paley $\Phi$-cotype $p'$.*

Part (1) of this proposition was proved in [9] (Proposition 7.7); the proof of part (2) is similar.

$B^*$ will denote the dual space of the Banach space $B$. We shall use the following lemma.

LEMMA 2.9. *Let $r,s\in(1,\infty)$ and $p\in(1,2]$. Then the following properties are equivalent:*

(1) *There exists a constant $A$ such that for every finite sequence $b=\{b_k\}$ $(b_k\in B)$*

$$\Big\|\sum_k b_k\varphi_k\Big\|_{L_B^{p',r}} \leqq A\|b\|_{l_B^{p,s}}. \tag{2.11}$$

(2) *There exists a constant $A'$ such that for every function $g\in L_{B^*}^{p,r'}$ the sequence $c(g)=\{c_n(g)\}$ of its Fourier coefficients satisfies the inequality*

$$\big\|c(g)\big\|_{l_{B^*}^{p',s'}} \leqq A'\|g\|_{L_{B^*}^{p,r'}}. \tag{2.12}$$

*The statement remains true if we exchange the rôles of $B$ and $B^*$.*

Setting $r=p'$, $s=p$, we obtain the following

THEOREM 2.10. *Let $1<p\leqq 2$. A Banach space $B$ is of Riesz $\Phi$-type $p$ (strong Riesz $\Phi$-cotype $p'$) if and only if its dual $B^*$ is of strong Riesz $\Phi$-cotype $p'$ (respectively, Riesz $\Phi$-type $p$).*

The proof of this theorem can be found in [9] (Theorem 7.8). Lemma 2.9 can be easily proved by similar standard arguments. We only have to use the following proposition.

PROPOSITION 2.11. *Let $B$ be a Banach space and assume $1<p,r<\infty$. Then there exists a constant $C_{p,r}$ such that for every function $g\in L_B^{p,r}$*

$$\|f\|_{L_B^{p,r}} \leqq C_{p,r}\sup\Big|\int_I \big\langle f(x),g(x)\big\rangle\,d\mu\Big|, \tag{2.13}$$

*where the supremum is taken over all functions* $g \in L_{B^*}^{p',r'}$ *with* $\|g\|_{L_{B^*}^{p',r'}} = 1$.

The proof of this statement also uses the standard technique. In the scalar case, inequality (2.13) is well known (see [2, p. 221]). Using this fact and the approximation by simple functions in $L_B^{p,r}$, one can easily obtain (2.13) in the vector-valued case. For $p = r$ the proof of (2.13) can be found in [8, p. 97].

Now set $r = s = p$, $1 < p < 2$ in Lemma 2.9. We obtain that if the space $B$ is of Paley $\Phi$-type $p$ (strong Paley $\Phi$-cotype $p'$), then its dual $B^*$ possesses a new property, expressed by inequalities of the form (1.6) (correspondingly, (1.5)). Thus it is natural to introduce the following definitions.

DEFINITION 2.12. Let $2 < q < \infty$. A Banach space $B$ is said to be of

(1) Paley $\Phi$-cotype* $q$, if there exists a constant $A$ such that for every polynomial $f = \sum_{k=1}^{n} b_k \varphi_k$ $(b_k \in B)$

$$\big\| \{b_k\} \big\|_{l_B^q} \leqq A \|f\|_{L_B^{q',q}}; \tag{2.14}$$

(2) strong Paley $\Phi$-cotype* $q$, if there exists a constant $A$ such that for every function $f \in L_B^1$ the sequence of its Fourier coefficients satisfies the inequality

$$\big\| c(f) \big\|_{l_B^q} \leqq A \|f\|_{L_B^{q',q}}, \quad c(f) = \big\{ c_k(f) \big\}. \tag{2.15}$$

We have the analogue of Proposition 2.8.

PROPOSITION 2.13. *Let* $2 < q < \infty$ *and suppose that* $\Phi$ *is a uniformly bounded system which is complete in* $L^{q',q}(I)$. *If a Banach space* $B$ *is of Paley* $\Phi$*-cotype** $q$, *then* $B$ *is of strong Paley* $\Phi$*-cotype** $q$.

Next, we give the following

DEFINITION 2.14. Let $2 < q < \infty$. We will say that a Banach space $B$ is of Paley $\Phi$-type* $q'$, if there exists a constant $A$ such that for every finite sequence $b = \{b_k\}_{k=1}^{n}$ $(b_k \in B)$

$$\|f\|_{L_B^q} \leqq A \|b\|_{l_B^{q',q}}, \quad \text{where} \quad f = \sum_{k=1}^{n} b_k \varphi_k. \tag{2.16}$$

REMARK 2.15. If $B$ is of Paley $\Phi$-type* $q'$ $(2 < q < \infty)$, then for every sequence $b = \{b_k\}_{k=1}^{\infty} \in l_B^{q',q}$ the series $\sum_{k=1}^{\infty} b_k \varphi_k$ converges in $L_B^q$ and its sum $f$ satisfies the inequality (2.16).

REMARK 2.16. For $2 < q < \infty$ we have (see [2, p. 217])

$$\|f\|_{L_B^{q',q}} \leqq C_q \|f\|_{L_B^{q'}}.$$

Thus, if a Banach space is of (strong) Paley $\Phi$-cotype* $q$, then it is also of (strong) Riesz $\Phi$-cotype $q$. Analogously, Paley $\Phi$-type* $q'$ of a Banach space $B$ implies Riesz $\Phi$-type $q'$ of $B$.

We shall see in §4 that the property of being Paley $\Phi$-type $p$ and that of being Paley $\Phi$-type* $p$ are not related to each other in the sense that none of them is stronger than the other.

Lemma 2.9 contains the following duality theorem.

THEOREM 2.17. *Let $B$ be a Banach space and $1 < p < 2$. Then:*

(1) *$B$ is of Paley $\Phi$-type $p$ if and only if $B^*$ is of strong Paley $\Phi$-cotype* $p'$;*

(2) *$B$ is of strong Paley $\Phi$-cotype $p'$ if and only if $B^*$ is of Paley $\Phi$-type* $p$.*

We also have the following properties of type and cotype.

THEOREM 2.18. *Let a Banach space $B$ be of Riesz $\Phi$-type $p$, $1 < p \leqq 2$. Then, for each $1 < r < p$, $B$ is of Paley $\Phi$-type $r$ and Paley $\Phi$-type* $r$. Moreover, for every $r \in (1,p)$ and every $s > 0$, there exists a constant $K$ such that for every finite sequence $b = \{b_k\}$ $(b_k \in B)$*

$$\|f\|_{L_B^{r',s}} \leqq K\|b\|_{l_B^{r,s}}, \quad \text{where} \quad f = \sum_k b_k\varphi_k. \tag{2.17}$$

THEOREM 2.19. *Let a Banach space $B$ be of strong Riesz $\Phi$-cotype $p'$, $1 < p \leqq 2$. Then $B$ is of strong Paley $\Phi$-cotype $r'$ and strong Paley $\Phi$-cotype* $r'$ for every $r$ such that $1 < r < p$. Moreover, for every $r \in (1,p)$ and every $s > 0$ there exists a constant $K$, such that for each function $f \in L_B^{r,s}$*

$$\big\|c(f)\big\|_{l_B^{r',s}} \leqq K\|f\|_{L_B^{r,s}}, \quad \text{where} \quad c(f) = \big\{c_n(f)\big\}. \tag{2.18}$$

Actually Theorems 2.18 and 2.19 were proved in [9] (Theorem 7.11) by interpolation methods. It was also pointed out in [9] that these results can be obtained directly from the corresponding estimates of the rearrangements.

## §3. On some concrete systems

The notions introduced in §2 are based on the theorems by Riesz and Paley (see §1). In both theorems there is a symmetry between the sum of an orthogonal series and the sequence of its Fourier coefficients. This symmetry may break down for a series with vector-valued coefficients. In other words, $\Phi$-type $p$ and $\Phi$-cotype $p'$ may represent quite different properties of the Banach spaces (see [16], p. 72).

We shall show that, for the trigonometric system, being of a given type and being of the corresponding (conjugate) cotype, are equivalent properties. The same result holds for the Walsh system.

LEMMA 3.1. *Let $B$ be a Banach space, $1 < p \leqq 2$ and $1 < r, s < \infty$. Then the following properties are equivalent:*

(i) *there exists a constant $A$ such that for every trigonometric polynomial*

$$f(x) = \sum_{k=-n}^{n} b_k e^{2\pi i k x} \quad \textit{with} \quad b \equiv \{b_k\}_{k=-n}^{n} \subset B \tag{3.1}$$

*the inequality*

$$\|f\|_{L_B^{p',r}} \leqq A \|b\|_{l_B^{p,s}} \tag{3.2}$$

*holds;*

(ii) *there exists a constant $A'$ such that for every polynomial $f$ as in* (3.1), *the inequality*

$$\|b\|_{l_B^{p',r}} \leqq A' \|f\|_{L_B^{p,s}} \tag{3.3}$$

*holds.*

PROOF. Let us fix the polynomial (3.1). The function $\|f(x)\|$ is 1-periodic on $\mathbf{R}$. Let $\varepsilon > 0$. There exists $\delta = 1/N$ ($N > 2n$ and $N$ even) such that on every interval of length $\delta$ the oscillation of the function $\|f(x)\|$ is less than $\varepsilon$.

Set $\Delta_k = \big[k\delta, (k+1)\delta\big)$ $(k \in \mathbf{Z})$. Let

$$g(x) = \sum_{k=0}^{2n} b_{n-k} \chi_{\Delta_k}(x) \quad \text{for} \quad 0 \leqq x < 1$$

and then consider the function $g$ extended to the whole line $\mathbf{R}$ with period 1. Clearly

$$\|g(x)\| = \sum_{k=0}^{2n} \|b_{n-k}\| \chi_{\Delta_k}(x)$$

and

$$g^*(t) = \sum_{k=0}^{2n} b_k^* \chi_{\Delta_k}(t), \quad 0 \leqq t < 1, \tag{3.4}$$

where $\{b_k^*\}_{k=0}^{2n}$ is the non-increasing rearrangement of the sequence $\{\|b_j\|_{j=-n}^n\}$.

Furthermore, for every $\nu \in \mathbf{Z}$ $(\nu \neq 0)$

$$c_\nu \equiv c_\nu(g) = \int_0^1 g(x) e^{-2\pi i\nu x}\,dx = \sum_{k=0}^{2n} b_{n-k} \int_{\Delta_k} e^{-2\pi i\nu x}\,dx$$

$$= \frac{1-e^{-2\pi i\nu\delta}}{2\pi i\nu} e^{-2\pi i n\nu\delta} f(\nu\delta),$$

and $c_0 \equiv c_0(g) = \delta f(0)$. Therefore,

(3.5)

$$\|c_{\nu+jN}\| = \frac{|\sin \pi\nu\delta|}{\pi|\nu+jN|}\big\| f(\nu\delta)\big\| \quad (\nu+jN \neq 0,\ -N/2 \leqq \nu < N/2,\ j \in \mathbf{Z}).$$

Let $\{c_k^*\}_{k=1}^\infty$ and $\{f_k^*\}_{k=1}^N$ be the non-increasing rearrangements of the sequences $\{\|c_\nu\|\}_{\nu=-\infty}^{\infty}$ and $\{\|f(\nu\delta)\|\}_{\nu=-N/2}^{N/2-1}$ respectively. Since (see (3.5))

$$\|c_\nu\| \geqq \frac{2}{\pi}\delta\big\| f(\nu\delta)\big\|, \qquad -N/2 \leqq \nu < N/2,$$

then

$$f_k^* \leqq \frac{\pi}{2\delta} c_k^* \quad (k = 1, \ldots N). \tag{3.6}$$

On the other hand,

$$\|c_{\nu+jN}\| \leqq \left|\frac{\nu}{\nu+jN}\right| \big\| f(\nu\delta)\big\|\,\delta \leqq \frac{2}{|j|+1}\big\| f(\nu\delta)\big|\,\delta \tag{3.7}$$

$(-N/2 \leqq \nu < N/2,\ \nu+jN \neq 0,\ j \in \mathbf{Z})$. Let $1 < q < \min(s,p)$. Then, by (3.7),

$$(c_k^*)^q \leqq \frac{1}{k}\sum_{j=1}^k (c_j^*)^q = \frac{1}{k} \sup_{\Lambda\subset\mathbf{Z},|\Lambda|=k} \sum_{m\in\Lambda} \|c_m\|^q \tag{3.8}$$

$$\leqq A_q \frac{\delta^q}{k} \sum_{j=1}^{\min(k,N)} (f_j^*)^q, \quad k \in \mathbf{N}.$$

Now suppose that we have the property (i). Since $g$ is a simple function, we know that its Fourier series converges to $g$ in the norm of $L_B^{p',r}$. Therefore, by our assumption,

$$\|g\|_{L_B^{p',r}} \leqq A \big\| \{c_\nu\} \big\|_{l_B^{p,s}}. \tag{3.9}$$

By using the estimate (3.8), taking also into account that $q < p$, we have

$$\big\| \{c_\nu\} \big\|_{l_B^{p,s}} \tag{3.10}$$

$$\leqq C\delta \left[ \sum_{k=1}^{N} k^{s/p-s/q-1} \left( \sum_{j=1}^{k} (f_j^*)^q \right)^{s/q} + \left( \sum_{k=1}^{N} (f_k^*)^q \right)^{s/q} N^{s/p-s/q} \right]^{1/s}.$$

Now, applying Hölder's inequality with exponents $s/q$ $(q < s)$ and its conjugate to the last sum, we get

$$\left( \sum_{k=1}^{N} (f_k^*)^q \right)^{s/q} \leqq C N^{s/q-s/p} \sum_{k=1}^{N} k^{s/p-1} (f_k^*)^s.$$

Also, by applying Hardy's inequality to the first sum in the right hand side of (3.10), we obtain

$$\big\| \{c_\nu\} \big\|_{l_B^{p,s}} \leqq C\delta \left( \sum_{k=1}^{N} k^{s/p-1} (f_k^*)^s \right)^{1/s}. \tag{3.11}$$

Denote $\overline{f}(x) = \sum_{k=-N/2}^{N/2-1} f(k\delta) \chi_{\Delta_k}(x)$. Then, by the choice of $\delta$, we have

$$\big| \|f(x)\| - \|\overline{f}(x)\| \big| \leqq \varepsilon, \quad x \in [-1/2, 1/2),$$

and

$$\overline{f}^*(t) - \varepsilon \leqq f^*(t) \leqq \overline{f}^*(t) + \varepsilon, \qquad 0 \leqq t < 1. \tag{3.12}$$

Furthermore,

$$\overline{f}^*(t) = \sum_{k=0}^{N-1} f_{k+1}^* \chi_{\Delta_k}(t), \qquad 0 \leqq t < 1. \tag{3.13}$$

Thus,

$$\Big(\sum_{k=1}^{N} k^{s/p-1}(f_k^*)^s\Big)^{1/s} \leqq C\delta^{-1/p}\Big(\int_0^1 t^{s/p-1}\big(\overline{f}^*(t)\big)^s\,dt\Big)^{1/s}$$
$$\leqq C\delta^{-1/p}\Big(\int_0^1 t^{s/p-1}\big(f^*(t)+\varepsilon\big)^s\,dt\Big)^{1/s},$$

and, by virtue of (3.11),

$$\big\|\{c_\nu\}\big\|_{l_B^{p,s}} \leqq C\delta^{1/p'}\Big(\int_0^1 t^{s/p-1}\big(f^*(t)+\varepsilon\big)^s\,dt\Big)^{1/s}. \tag{3.14}$$

Next (see (3.4)),

$$\|g\|_{L_B^{p',r}} \geqq \frac{\delta^{1/p'}}{2}\Big(\sum_{k=0}^{2n}(k+1)^{r/p'-1}(b_k^*)^r\Big)^{1/r}. \tag{3.15}$$

By using (3.9), (3.14) and (3.15), we obtain

$$\|b\|_{l_B^{p',r}} \leqq A'\Big(\int_0^1 t^{s/p-1}\big(f^*(t)+\varepsilon\big)^s\,dt\Big)^{1/s}.$$

Hence inequality (3.3) follows at once.

Let us assume now that property (ii) holds. By applying this property to the partial sums of the Fourier series of the function $g$ and taking into account that this series converges to $g$ in the norm of $L_B^{p,s}$, we obtain the inequality

$$\big\|\{c_\nu\}\big\|_{l_B^{p',r}} \leqq A'\|g\|_{L_B^{p,s}}. \tag{3.16}$$

By using (3.12), (3.13) and (3.6), we have

$$\|f\|_{L_B^{p',r}} \leqq \Big(\int_0^1 t^{r/p'-1}\big(\overline{f}^*(t)+\varepsilon\big)^r\,dt\Big)^{1/r}$$
$$\leqq C\Big[\delta^{1/p'}\Big(\sum_{k=1}^{N} k^{r/p'-1}(f_k^*)^r\Big)^{1/r}+\varepsilon\Big]$$
$$\leqq C'\Big[\delta^{-1/p}\Big(\sum_{k=1}^{N} k^{r/p'-1}(c_k^*)^r\Big)^{1/r}+\varepsilon\Big].$$

On the other hand (see (3.4)),

$$\|g\|_{L_B^{p,s}} \leqq C\delta^{1/p}\left(\sum_{k=0}^{2n} k^{s/p-1}(b_k^*)^s\right)^{1/s}.$$

Thus, by using (3.16), we obtain

$$\|f\|_{L_B^{p',r}} \leqq A\big(\|b\|_{l_B^{p,s}} + \varepsilon\big).$$

This implies (3.2) and the lemma is proved. □

The exact counterpart of Lemma 3.1 is also true for the system of Walsh functions $W = \{w_n\}_{n=0}^{\infty}$ (see [9] for the definition). For $p = r = s$ this was proved in [9]; the proof for the general case is similar. Thus, we have, denoting $T = \{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ (and applying also Propositions 2.8 and 2.9)

THEOREM 3.2. *Let $B$ be a Banach space, $1 < p \leqq 2$, and $\Phi = T$ or $\Phi = W$. Then:*

(1) *$B$ is of Riesz $\Phi$-type $p$ if and only if $B$ is of strong Riesz $\Phi$-cotype $p'$;*

(2) *$B$ is of Paley $\Phi$-type $p$ if and only if $B$ is of strong Paley $\Phi$-cotype $p'$;*

(3) *$B$ is of Paley $\Phi$-type* $p$ if and only if $B$ is of strong Paley $\Phi$-cotype* $p'$.*

Part (1) of this theorem was proved in [9] (Theorems 7.13 and 7.14). Related results for Fourier type were obtained in [1], [4], [6], [10] and [15].

Denote by $R$ the Rademacher system of functions

$$r_n(x) = \operatorname{sgn}\sin 2^n\pi x \qquad \big(x \in [0,1], n \in \mathbf{N}\big).$$

Note that Riesz $R$-type and Riesz $R$-cotype are usually called simply type and cotype.

Applying the Kahane's inequality (see, for example, [13]), we immediately obtain

PROPOSITION 3.3. *Let $B$ be a Banach space. Then:*

(i) *$B$ is of Riesz $R$-type $p$ $(1 < p < 2)$ if and only if $B$ is of Paley $R$-type $p$;*

(ii) *$B$ is of Riesz $R$-cotype $q$ $(2 < q < \infty)$ if and only if $B$ is of Paley $R$-cotype* $q$;*

(iii) *if $B$ is of Paley $R$-type* $p$, then $B$ is of Paley $R$-type $p$ $(1 < p < 2)$;*

(iv) *if $B$ is of Paley $R$-cotype $q$, then $B$ is of Paley $R$-cotype* $q$ $(2 < q < \infty)$.*

In §4 this proposition will be complemented by counterexamples which show that the statements (iii) and (iv) cannot be reversed.

It is known that if $B$ is of Fourier type $p$ $(1<p<2)$, then $B$ is of type $p$ and cotype $p'$ (see [1], [9], [10], [14]). We have a more general statement.

Let $(I,\mu)$ be a measure space $(\mu(I)=1)$.

THEOREM 3.4. *Let $B$ be a Banach space and let $\Phi$ be a uniformly bounded orthonormal system on $I$. Then:*

(i) *if $B$ is of Riesz $\Phi$-type $p$ $(1<p\leqq 2)$, it follows that $B$ is of Riesz $R$-type $p$;*

(ii) *if $B$ is of Riesz $\Phi$-cotype $q$ $(2\leqq q<\infty)$, it follows that $B$ is of Riesz $R$-cotype $q$.*

This theorem was proved in [9]; the proof made use of the contraction principle. Similarly we obtain the following theorem.

THEOREM 3.5. *Let $B$ be a Banach space and let $\Phi$ be a uniformly bounded orthonormal system on $I$. Then:*

(i) *if $B$ is of Paley $\Phi$-type* $p$ $(1<p\leqq 2)$, it follows that $B$ is of Paley $R$-type* $p$;*

(ii) *if $B$ is of Paley $\Phi$-cotype $q$ $(2\leqq q<\infty)$, it follows that $B$ is of Paley $R$-cotype $q$.*

Of course, Paley $\Phi$-type $p$ implies Paley $R$-type $p$, and Paley $\Phi$-cotype* $q$ implies Paley $R$-cotype* $q$. This follows from Theorem 3.4 and Proposition 3.3.

## §4. Types and cotypes of $L^p$-spaces

Let $(I,\mu)$ be a non-atomic measure space $(\mu(I)=1)$ and $\Phi$ be an orthonormal system on $I$ satisfying the condition (2.2). It is known that the space $L^p(\Omega,\nu)$ (where $1<p<\infty$ and $(\Omega,\nu)$ is an arbitrary measure space) is of Riesz $\Phi$-type $\min(p,p')$ and in the general case this result cannot be improved (see [9]). Now we will prove some stronger assertions.

THEOREM 4.1. *Let $2<q<\infty$ and $B=L^q(\Omega,\nu)$. Then $B$ is of strong Paley $\Phi$-cotype* $q$ and Paley $\Phi$-type* $q'$.*

PROOF. Let $f$ be a simple function of the form

$$f(x)=\sum_{j=1}^{N} d_j\chi_{E_j}(x), \tag{4.1}$$

where $d_j\in B$ and $E_j\subset I$ are mutually disjoint $\mu$-measurable sets with $\mu(E_j)=1/N(j=1,\dots,N)$. Since the space $(I,\mu)$ is non-atomic, the set of all functions of the form (4.1) is dense in $L_B^{q',q}$.

We have

$$\sum_{k=1}^{\infty}\|c_k(f)\|^q=\sum_{k=1}^{\infty}\int_\Omega|c_k(f)(y)|^q d\nu(y). \tag{4.2}$$

Next, by inequality (1.6), for every $y\in\Omega$

$$\sum_{k=1}^{\infty}|c_k(f)(y)|^q\leqq A_q M^{q-2}\int_0^1\xi^{q-2}\big(f_y^*(\xi)\big)^q d\xi, \tag{4.3}$$

where $f_y^*(\xi)$ is the non-increasing rearrangement of the function

$$f_y(x)=f(x)(y)=\sum_{j=1}^{N}d_j(y)\chi_{E_j}(x)\qquad (x\in I;\ y\in\Omega \text{ is fixed}).$$

Let $\{\overline{d}_j(y)\}_{j=1}^N$ be the non-increasing rearrangement of the sequence $\{d_j(y)\}_{j=1}^N$. Then

$$f_y^*(\xi)=\sum_{j=1}^{N}\overline{d}_j(y)\chi_{\Delta_j}(\xi),\qquad \Delta_j=\left[\frac{j-1}{N},\frac{j}{N}\right),$$

and the integral on the right-hand side of (4.3) does not exceed

$$N^{1-q}\sum_{j=1}^{N}j^{q-2}\big(\overline{d}_j(y)\big)^q. \tag{4.4}$$

Now let $\{n_1,\dots,n_N\}$ be the permutation of the indices $\{1,\dots,N\}$ such that $\|d_{n_j}\|=d_j^*$ $(j=1,\dots,N)$. By the Hardy–Littlewood theorem (see [2, p. 44]), for every $y\in\Omega$

$$\sum_{j=1}^{N}j^{q-2}\big(\overline{d}_j(y)\big)^q\leqq\sum_{j=1}^{N}j^{q-2}|d_{n_j}(y)|^q. \tag{4.5}$$

Thus, we obtain (see (4.2)–(4.5))

$$\sum_{k=1}^{\infty}\|c_k(f)\|^q\leqq A_q N^{1-q}M^{q-2}\sum_{j=1}^{N}j^{q-2}\int_\Omega|d_{n_j}(y)|^q\,d\nu(y)$$

$$= A_q N^{1-q} M^{q-2} \sum_{j=1}^{N} j^{q-2} (d_j^*)^q.$$

But

$$\| f(x) \| = \sum_{j=1}^{N} \|d_j\| \chi_{E_j}(x) \quad \text{and} \quad f^*(\xi) = \sum_{j=1}^{N} d_j^* \chi_{\Delta_j}(\xi).$$

Thus,

$$\sum_{k=1}^{\infty} \| c_k(f) \|^q \leqq A_q' M^{q-2} \int_0^1 \xi^{q-2} \big( f^*(\xi) \big)^q d\xi$$

and therefore $B$ is of strong Paley $\Phi$-cotype* $q$. Similarly we obtain that $B$ is of Paley $\Phi$-type* $q'$. $\square$

Using Theorem 2.17, we obtain

THEOREM 4.2. *Let $1 < p < 2$ and $B = L^p(\Omega, \nu)$. Then $B$ is of Paley $\Phi$-type $p$ and strong Paley $\Phi$-cotype $p'$.*

The values of the types and cotypes in Theorems 4.1 and 4.2 are sharp. This follows from the corresponding statements for the Riesz types and cotypes (see [9]).

Now we will obtain some negative results. First of all we consider the following general construction.

LEMMA 4.3. *Suppose that there are given:*

(1) *some numbers $p, r, q, s \in (1, \infty)$;*

(2) *a positive decreasing sequence $\{\eta_k\}$ with $\eta_1 = 1$, $\eta_k > \eta_{k+1}$, $\lim_{k\to\infty} \eta_k = 0$;*

(3) *a positive increasing sequence $\{a_k\}$ such that $\{ a_k(\eta_k - \eta_{k+1})^{1/p} \}$ decreases;*

(4) *an orthonormal system $\Psi = \{\psi_n\}$ on $[0,1]$ with $|\psi_n(x)| = 1$ almost everywhere $(n \in \mathbf{N})$.*

*Furthermore, let $b_k(y) = a_k \chi_{\Delta_k}(y)$, $\Delta_k = (\eta_{k+1}, \eta_k]$, and*

$$F(x,y) = \sum_{k=1}^{\infty} b_k(y) \psi_k(x). \tag{4.6}$$

*Then*

$$\| \{b_k\} \|_{l^{q,s}(L^{p,r})} = A_{p,r} \left( \sum_{k=1}^{\infty} k^{s/q-1} \big( a_k |\Delta_k|^{1/p} \big)^s \right)^{1/s}, \tag{4.7}$$

*and for almost all* $x \in [0,1]$

$$\left\| F(x,\cdot)\right\|_{L^{p,r}} = A_{p,r}\left( \sum_{k=1}^{\infty} a_k^r \left( \eta_k^{r/p} - \eta_{k+1}^{r/p} \right) \right)^{1/r}. \tag{4.8}$$

PROOF. We have

$$\|b_k\|_{L^{p,r}} = \left(\frac{p}{r}\right)^{1/r} a_k |\Delta_k|^{1/p}, \qquad k \in \mathbf{N}.$$

Since this sequence decreases, we get (4.7).

Next, for every $y \in \Delta_k$ and almost all $x \in [0,1]$ $\left|F(x,y)\right| = a_k$. As $\{a_k\}$ increases, we get

$$\left\| F(x,\cdot)\right\|_{L^{p,r}} = \left( \sum_{k=1}^{\infty} a_k^r \int_{\eta_{k+1}}^{\eta_k} t^{r/p-1}\,dt \right)^{1/r}$$

$$= \left(\frac{p}{r}\right)^{1/r} \left( \sum_{k=1}^{\infty} a_k^r \left( \eta_k^{r/p} - \eta_{k+1}^{r/p} \right) \right)^{1/r}. \qquad \square$$

PROPOSITION 4.4. *The space* $B = L^p[0,1]$ $(1 < p < 2)$ *is not of Paley* $\Phi$*-type** $p$ *for any uniformly bounded ONS* $\Phi$*.*

PROOF. Let $a_k = (\ln k + 1)^{1/p}$, $\eta_k = (\ln k + 1)^{-1}$ $(k \in \mathbf{N})$. Then the sequence $a_k(\eta_k - \eta_{k+1})^{1/p} \sim \left[k(\ln k + 1)\right]^{-1/p}$ decreases. Let us apply Lemma 4.3 with $q = r = p$, $s = p'$ and $\Psi = R$. In our case the series (4.7) is

$$\sum_{k=1}^{\infty} \frac{1}{k(\ln k + 1)^{p'/p}} < \frac{1}{2-p}, \tag{4.9}$$

i.e. convergent, while the series (4.8)

$$\sum_{k=1}^{\infty} \frac{1}{k(\ln k + 1)} = \infty \tag{4.10}$$

diverges. Let $f_n(x) = \sum_{k=1}^{n} b_k r_k(x)$. It is easy to see that for every $x \in [0,1]$

$$\lim_{n\to\infty} \left\| f_n(x)\right\|_B = \left\| F(x,\cdot)\right\|_B$$

(see (4.6)). But the right hand side is infinite for almost all $x \in [0,1]$ (see (4.8) and (4.10)). Therefore,

$$\lim_{n\to\infty} \left\| f_n\right\|_{L_B^{p'}} = \infty,$$

while

$$\left\| \{b_k\}_{k=1}^n \right\|_{l_B^{p,p'}} \leqq A'_p < \infty$$

(see (4.7) and (4.9)). Thus, $L^p$ is not of Paley $R$-type* $p$. Applying Theorem 3.5, we complete the proof. $\square$

As above, we denote by $T$ the trigonometric system, and by $W$ the Walsh system.

PROPOSITION 4.5. *The space* $L^p[0,1]$ $(1 < p < 2)$ *is not of Paley* $\Phi$*-cotype** $p'$ *neither for* $\Phi = T$ *nor for* $\Phi = W$.

This is an immediate consequence of the Proposition 4.4 and Theorem 3.2 (see also Proposition 2.13).

PROPOSITION 4.6. *The space* $B = L^q[0,1]$ $(2 < q < \infty)$ *is not of Paley* $\Phi$*-cotype* $q$ *for any uniformly bounded ONS* $\Phi$.

PROOF. We use Lemma 4.3, setting $s = q'$, $p = r = q$, $a_k = k^{1/q}(\ln k + 1)^{-1/q'}$, $\eta_k = 1/k$ $(k \in \mathbf{N})$ and $\Psi = R$. The series (4.7) assumes the form

$$\sum_{k=1}^{\infty} \frac{1}{k(\ln k + 1)} = \infty.$$

On the other hand, the series (4.8)

$$\sum_{k=1}^{\infty} \frac{1}{k(\ln k + 1)^{q-1}}$$

converges.

Let $f_n(x) = \sum_{k=1}^n b_k r_k(x)$. Then

$$\left\| \{b_k\}_{k=1}^n \right\|_{l_B^{q,q'}} \to \infty \quad \text{as} \quad n \to \infty,$$

and the sequence $\left\{ \|f_n\|_{L_B^\infty} \right\}$ is bounded. It follows that $B$ does not have Paley $R$-cotype $q$. It remains to apply Theorem 3.5. $\square$

Using Theorem 3.2, we also get

PROPOSITION 4.7. *The space* $B = L^q[0,1]$ $(2 < q < \infty)$ *is not of Paley* $\Phi$*-type* $q'$ *for neither* $\Phi = T$ *nor* $\Phi = W$.

From Theorems 4.1 and 4.2 and Propositions 4.4 and 4.6 we deduce the following complement to Proposition 3.3.

COROLLARY 4.8. *For every uniformly bounded ONS* $\Phi$,
(1) *Paley* $\Phi$*-type* $p$ $(1<p<2)$ *does not imply Paley* $R$*-type** $p$;
(2) *strong Paley* $\Phi$*-cotype** $q$ $(2<q<\infty)$ *does not imply Paley* $R$*-cotype* $q$.

Besides, we obtain the following general statement.

COROLLARY 4.9. *Let* $\Phi = T$ *or* $\Phi = W$. *Then:*
(1) *A Banach space may be of Paley* $\Phi$*-type* $p$ $(1<p<2)$, *but not of Paley* $\Phi$*-type** $p$, *and vice versa;*
(2) *A Banach space may be of Paley* $\Phi$*-cotype* $q$ $(2<q<\infty)$, *but not of Paley* $\Phi$*-cotype** $q$, *and vice versa.*

## §5. Interpolation of Paley type spaces

Let $\{B_0, B_1\}$ be an interpolation couple of Banach spaces. If $0<\theta<1$ and $1 \leqq r < \infty$, then we shall denote by $(B_0, B_1)_{\theta,r} \equiv B_{\theta,r}$ the interpolation space obtained by the real interpolation method. Also, we shall denote by $[B_0, B_1]_\theta \equiv B_\theta$ the interpolation space obtained by the complex interpolation method. See [3] and [20].

It was proved by J. Peetre [18] that if $B_i$ are of Fourier types $p_i \in [1,2]$ $(i=0,1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, then $B_{\theta,p}$ and $B_\theta$ both are of Fourier type $p$. Furthermore, H. König [14] showed that for every $p \leqq r \leqq p'$, the space $B_{\theta,r}$ is of Fourier type $p$ (see also [9]).

Note that the complex interpolation of spaces with given Fourier types has also been considered in the work of M. Cwikel and Y. Sagher [7].

In this section we shall study the behaviour under interpolation of the Paley types with respect to general ONS. As applications of these theorems, we shall obtain sharp coefficient estimates for functions with values in the Lorentz space $L^{p,r}$.

Sometimes we will use the notation $L^{p,r}(B)$ instead of $L^{p,r}_B$.

LEMMA 5.1 [14]. *Let* $1<p_0,p_1 \leqq 2$, $0<\theta<1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $p \leqq r \leqq p'$. *Then, for any interpolation couple of Banach spaces* $\{B_0, B_1\}$, *we have the following continuous embedding maps*

$$L^p(B_{\theta,r}) \hookrightarrow \big(L^{p_0}(B_0), L^{p_1}(B_1)\big)_{\theta,r}, \tag{5.1}$$

$$\big(L^{p_0'}(B_0), L^{p_1'}(B_1)\big)_{\theta,r} \hookrightarrow L^{p'}(B_{\theta,r}), \tag{5.2}$$

$$l^p(B_{\theta,r}) \hookrightarrow \big(l^{p_0}(B_0), l^{p_1}(B_1)\big)_{\theta,r}, \tag{5.3}$$

$$\big(l^{p_0'}(B_0), l^{p_1'}(B_1)\big)_{\theta,r} \hookrightarrow l^{p'}(B_{\theta,r}). \tag{5.4}$$

Let $\Phi = \{\varphi_n\}$ be a uniformly bounded orthonormal system of functions on a space $I$ with a non-atomic measure $\mu$ $\big(\mu(I) = 1\big)$.

THEOREM 5.2. *Let $\{B_0, B_1\}$ be an interpolation couple of Banach spaces such that $B_i$ is of Paley $\Phi$-type $p_i$ $(i = 0, 1;\ 1 < p_0 \leqq p_1 < 2)$. Suppose that $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $p \leqq r \leqq p'$. Then there exists a constant $A$ such that for every finite sequence $b = \{b_k\}_{k=1}^n$ $(b_k \in B \equiv B_{\theta,r})$*

$$\|f\|_{L_B^{p',r}} \leqq A\|b\|_{l_B^p}, \quad \textit{where} \quad f = \sum_{k=1}^n b_k\varphi_k. \tag{5.5}$$

PROOF. If $r = p'$ then (5.5) follows from the embeddings (5.2) and (5.3). Let $r < p'$. Moreover, suppose first that

$$r < p/(p_1 - 1). \tag{5.6}$$

Using the $L$-method (see [18, p. 29]), we choose a representation $b = b^0(t) + b^1(t)$, $0 < t < \infty$, $b^i(t) = \big\{ b_k^i(t) \big\}_{k=1}^n$, $b_k^{(i)}(t) \in B_i$ $(i = 0, 1)$, so that

$$\begin{aligned} &\|b\|^r_{(l^{p_0}(B_0), l^{p_1}(B_1))_{\theta,r}} \\ &\geqq C_1 \int_0^\infty \Big[ t^{-\eta}\big(\big\| b^0(t)\big\|^{p_0}_{l^{p_0}(B_0)} + t\big\| b^1(t)\big\|^{p_1}_{l^{p_1}(B_1)}\big)\Big]^{r/p} \frac{dt}{t}, \end{aligned} \tag{5.7}$$

where $\eta = \theta p/p_1$ and $C_1$ is a positive constant depending on the given parameters but independent of $b$. Denote

$$f_t^i(x) = \sum_{k=1}^n b_k^i(t)\varphi_k(x) \qquad (i = 0, 1).$$

Then $f(x) = f_t^0(x) + f_t^1(x)$, $0 < t < \infty$. Using again the $L$-method, we have, for each $\lambda > 0$ and each $x \in I$:

$$\big\| f(x)\big\|^r_B \leqq C \int_0^\infty \Big[ t^{-\eta}\big(\big\| f^0_{\lambda t}(x)\big\|^{p_0}_{B_0} + t\big\| f^1_{\lambda t}(x)\big\|^{p_1}_{B_1}\big)\Big]^{r/p} \frac{dt}{t}$$

$$\leqq C' \max\Big[\lambda^{\eta r/p} \int_0^\infty \big(t^{-\eta}\big\| f^0_t(x)\big\|^{p_0}_{B_0}\big)^{r/p}\frac{dt}{t}, \lambda^{(\eta-1)r/p}$$

$$\int_0^\infty \big(t^{1-\eta}\big\| f^1_t(x)\big\|^{p_1}_{B_1}\big)^{r/p}\frac{dt}{t}\Big] \equiv C' \max\big[\lambda^{\eta r/p}J_0(x), \lambda^{(\eta-1)r/p}J_1(x)\big].$$

Thus, with an appropriate choice of $\lambda$, we have

$$\| f(x)\|_B^r \leqq C\big(J_0(x)\big)^{1-\eta}\big(J_1(x)\big)^{\eta}. \tag{5.8}$$

Next, since the space $(I,\mu)$ is non-atomic,

$$\begin{aligned} K \equiv \|f\|^r_{L_B^{p',r}} &\leqq \int_0^1 \xi^{r/p'-2}\,d\xi \int_0^\xi \big(f^*(u)\big)^r\,du \\ &= \int_0^1 \xi^{r/p'-2}\,d\xi \int_{E_\xi} \| f(x)\|_B^r\,d\mu, \end{aligned} \tag{5.9}$$

where $E_\xi \subset I$ is some measurable set with $\mu(E_\xi)=\xi$ (see [2, p. 46]).

Let $s_i = rp_i/p$; then $s_i \geqq p_i$ $(i=0,1)$. It is easy to see that $p=(1-\eta)p_0+\eta p_1$, and therefore

$$\frac{r}{p'} = \frac{s_0}{p_0'}(1-\eta)+\frac{s_1}{p_1'}\eta.$$

By using (5.9), (5.8) and Hölder's inequality, we obtain

$$K \leqq C\int_0^1 \xi^{r/p'-2}\left(\int_{E_\xi} J_0(x)\,d\mu\right)^{1-\eta}\left(\int_{E_\xi} J_1(x)\,d\mu\right)^{\eta} d\xi \leqq CK_0^{1-\eta}K_1^{\eta},$$

where

$$K_i = \int_0^1 \xi^{s_i/p_i'-2}\,d\xi\int_{E_\xi} J_i(x)\,d\mu, \quad i=0,1.$$

In view of (5.6), $s_i < p_i'$; besides, $s_i \geqq p_i$. Thus, by applying Fubini theorem and taking into account that $B_i$ is of Paley $\Phi$-type $p_i$, we get, for every $t>0$

$$\begin{aligned} \int_0^1 \xi^{s_i/p_i'-2}\,d\xi\int_{E_\xi} \| f_t^i(x)\|_{B_i}^{s_i}\,d\mu &\leqq \int_0^1 \xi^{s_i/p_i'-2}\,d\xi\int_0^\xi \big((f_t^i)^*(u)\big)^{s_i}\,du \\ \leqq C\|f_t^i\|^{s_i}_{L^{p_i',s_i}(B_i)} \leqq C'\|f_t^i\|^{s_i}_{L^{p_i',p_i}(B_i)} &\leqq C''\| b^i(t)\|^{s_i}_{l^{p_i}(B_i)}, \quad i=0,1. \end{aligned}$$

Hence,

$$K_0 \leqq C\int_0^\infty \big[t^{-\eta}\| b^0(t)\|^{p_0}_{l^{p_0}(B_0)}\big]^{r/p}\,\frac{dt}{t}$$

and

$$K_1 \leqq C\int_0^\infty \big[t^{1-\eta}\| b^1(t)\|^{p_1}_{l^{p_1}(B_1)}\big]^{r/p}\,\frac{dt}{t}.$$

By using (5.7), we have

$$K \leqq CK_0^{1-\eta}K_1^{\eta} \leqq C(K_0+K_1) \leqq C'\|b\|^r_{(l^{p_0}(B_0),l^{p_1}(B_1))_{\theta,r}}.$$

Then, by applying (5.3), we obtain the inequality (5.5).

We have proved (5.5) assuming that (5.6) holds. Now we will show that the general case can be reduced to the one considered above with the help of the reiteration theorem.

Since $r < p'$, then

$$\frac{1}{p} = \frac{1}{p_0} - \theta\left(\frac{1}{p_0} - \frac{1}{p_1}\right) > \frac{1}{1+p/r}.$$

Therefore we can choose $\theta < \theta_1 < 1$, so that

$$\frac{1}{\overline{p}_1} \equiv \frac{1}{p_0} - \theta_1\left(\frac{1}{p_0} - \frac{1}{p_1}\right) > \frac{1}{1+p/r}. \tag{5.10}$$

We also choose an arbitrary $0 < \theta_0 < \theta$ and set

$$\frac{1}{\overline{p}_0} = \frac{1-\theta_0}{p_0} + \frac{\theta_0}{p_1}.$$

We have $\overline{p}_0 < p < \overline{p}_1$. Now choose $\gamma$ such that

$$\frac{1}{p} = \frac{1-\gamma}{\overline{p}_0} + \frac{\gamma}{\overline{p}_1}.$$

It is easy to see that $\theta = (1-\gamma)\theta_0 + \gamma\theta_1$.

Let $\overline{B}_i = (B_0, B_1)_{\theta_i,\overline{p}_i}$ $(i = 0, 1)$. Then, as it has been already proved, $\overline{B}_i$ is of Paley $\Phi$-type $\overline{p}_i$. By the reiteration theorem (see [3]),

$$(B_0, B_1)_{\theta,r} = (\overline{B}_0, \overline{B}_1)_{\gamma,r}. \tag{5.11}$$

But now we have the same case as above; namely, $r < p/(\overline{p}_1 - 1)$ (see (5.10)). Thus,

$$\|f\|_{L^{p',r}(\overline{B}_{\gamma,r})} \leqq A\|b\|_{l^p(\overline{B}_{\gamma,r})}.$$

Using (5.11), we obtain (5.5). $\square$

Corollary 5.3. *Let $B_i$ be of Paley $\Phi$-type $p_i$ $(i = 0, 1;\ 1 < p_0 \leqq p_1 < 2)$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then the space $(B_0, B_1)_{\theta,p}$ is of Paley $\Phi$-type $p$.*

THEOREM 5.4. *Let $(B_0, B_1)$ be an interpolation couple of Banach spaces such that $B_i$ are of strong Paley $\Phi$-cotype $p_i'$ $(i = 0, 1;\ 1 < p_0 \leqq p_1 < 2)$. Suppose that $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $p \leqq r \leqq p'$. Then there exists a constant $A$ such that for every $f \in L_B^p$ $(B = B_{\theta,r})$*

$$\left\| \{ c_n(f) \} \right\|_{l_B^{p',r}} \leqq A \|f\|_{L_B^p}. \tag{5.12}$$

The proof is the same as that of Theorem 5.2.

COROLLARY 5.5. *Let $B_i$ be of strong Paley $\Phi$-cotype $p_i'$ $(i = 0, 1;\ 1 < p_0, p_1 < 2)$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then the space $(B_0, B_1)_{\theta,p}$ is of strong Paley $\Phi$-cotype $p'$.*

THEOREM 5.6. *Let $\{B_0, B_1\}$ be an interpolation couple of Banach spaces such that $B_i$ is of Paley $\Phi$-type* $p_i$ $(i = 0, 1;\ 1 < p_0 \leqq p_1 < 2)$. Suppose that $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $p \leqq r \leqq p'$. Then there exists a constant $A$ such that for every finite sequence $b = \{b_k\}_{k=1}^n$, $b_k \in B \equiv B_{\theta,r}$,*

$$\|f\|_{L_B^{p'}} \leqq A \|b\|_{l_B^{p,r}}, \quad \textit{where} \quad f = \sum_{k=1}^n b_k \varphi_k. \tag{5.13}$$

PROOF. If $r = p$, then (5.13) follows from (5.2) and (5.3). Let $r > p$. Denote $q_i = p_i'$ $(i = 0, 1)$, $q = p'$. First suppose that

$$r > q/(q_1 - 1). \tag{5.14}$$

Let $\eta = \theta \frac{q}{q_1}$. Then $q = (1-\eta) q_0 + \eta q_1$. Using the $L$-method, for each $k = 1, \dots, n$, we choose a representation

$$b_k = \overline{b}_k^0(t) + \overline{b}_k^1(t) \qquad \big(0 < t < \infty,\ \overline{b}_k^i(t) \in B_i\big)$$

so that

$$\|b_k\|_B \geqq C_1 \left( \int_0^\infty \left[ t^{-\eta} \big( \big\| \overline{b}_k^0(t) \big\|_{B_0}^{q_0} + t \big\| \overline{b}_k^1(t) \big\|^{q_1} B_1 \big) \right]^{r/q} \frac{dt}{t} \right)^{1/r}, \tag{5.15}$$

where $C_1$ is a constant which depends on the given parameters but not on $b_k$.

Furthermore, we have

$$\|b\|_{l^{p,r}(B)}^r = \sum_{k=1}^n k^{r/p-1} (b_k^*)^r = \sum_{k=1}^n \sigma_k^{r/p-1} \|b_k\|_B^r,$$

where $\{\sigma_k\}_{k=1}^n$ is some rearrangement of the indices $\{1,\dots,n\}$. Denote

$$b_k^i(t)=\overline{b}_k^i(t\sigma_k^{-\gamma}),\qquad 0<t<\infty,\ i=0,1,$$

where $\gamma=q_0-q_1$. Then

$$b_k=b_k^0(t)+b_k^1(t),\qquad 0<t<\infty, \tag{5.16}$$

and (see (5.15))

$$\|b_k\|_B \geqq C_1\biggl(\int_0^\infty\Bigl[t^{-\eta}\bigl(\|b_k^0(\sigma_k^\gamma t)\|_{B_0}^{q_0}+t\|b_k^1(\sigma_k^\gamma t)\|_{B_1}^{q_1}\bigr)\Bigr]^{r/q}\frac{dt}{t}\biggr)^{1/r}.$$

Thus,

$$\begin{aligned}\|b\|_{l^{p,r}(B)}^r &\geqq C\biggl[\sum_{k=1}^n\sigma_k^{r/p-1}\int_0^\infty\bigl[t^{-\eta}\|b_k^0(\sigma_k^\gamma t)\|_{B_0}^{q_0}\bigr]^{r/q}\frac{dt}{t}\\ &\quad+\sum_{k=1}^n\sigma_k^{r/p-1}\int_0^\infty\bigl[t^{1-\eta}\|b_k^1(\sigma_k^\gamma t)\|_{B_1}^{q_1}\bigr]^{r/q}\frac{dt}{t}\biggr]\\ &=C\int_0^\infty z^{-\eta r/q}\biggl[\sum_{k=1}^n\sigma_k^{r/p-1+\gamma\eta r/q}\|b_k^0(z)\|_{B_0}^{q_0r/q}\\ &\quad+z^{r/q}\sum_{k=1}^n\sigma_k^{r/p-1+\gamma(\eta-1)r/q}\|b_k^1(z)\|_{B_1}^{q_1r/q}\biggr]\frac{dz}{z}.\end{aligned}$$

Denote $s_i=q_ir/q$ $(i=0,1)$; then $s_i\leqq q_i$. Next,

$$\frac{r}{p}+\frac{\gamma\eta r}{q}=\frac{s_0}{q_0'},\qquad \frac{r}{p}+\frac{\gamma(\eta-1)r}{q}=\frac{s_1}{q_1'}.$$

It follows from (5.14) that $s_i>q_i'$ $(i=0,1)$. By the Hardy–Littlewood inequality (see [3, p. 44]), for every $z>0$ and $i=0,1$

$$\sum_{k=1}^n\sigma_k^{s_i/q_i'-1}\|b_k^i(z)\|_{B_i}^{s_i}\geqq\sum_{k=1}^n k^{s_i/q_i'-1}\bigl((b^i(z))_k^*\bigr)^{s_i}, \tag{5.17}$$

where $b^i(z)=\{b_k^i(z)\}_{k=1}^n$ and $\{(b^i(z))_k^*\}_{k=1}^n$ is the non-increasing rearrangement of the sequence $\{\|b_k^i(z)\|_{B_i}\}_{k=1}^n$ (for a fixed $z$).

The sum on the right hand side of (5.17) equals

$$\left\| b_i(z)\right\|^{s_i}_{l^{q_i'},s_i(B_i)} \geqq C\left\| b^i(x)\right\|^{s_i}_{l^{q_i'},q_i(B_i)}$$

($s_i \leqq q_i$; $C$ is a positive constant). Thus, we obtain that

(5.18)

$$\|b\|^r_{l^{p,r}(B)} \geqq C\int_0^\infty \left[z^{-\eta}\left(\left\| b^0(z)\right\|^{q_0}_{l^{q_0'},q_0(B_0)} + z\left\| b^1(z)\right\|^{q_1}_{l^{q_1'},q_1(B_1)}\right)\right]^{r/q}\frac{dz}{z}.$$

Next, let

$$f^i_z(x) = \sum_{k=1}^n b^i_k(z)\varphi_k(x) \quad (i=0,1).$$

Then

$$f(x) = f^0_z(x) + f^1_z(x), \quad 0 < z < \infty. \tag{5.19}$$

Since $B_i$ is of Paley $\Phi$-type* $p_i = q_i'$, we have

$$\|f^i_z\|_{L^{q_i}(B_i)} \leqq C\left\| b^i(z)\right\|_{l^{q_i'},q_i(B_i)} \quad (0 < z < \infty). \tag{5.20}$$

Using (5.18)–(5.20) and applying the $L$-method we get

$$\begin{aligned}\|b\|^r_{l^{p,r}(B)} &\geqq C\int_0^\infty \left[z^{-\eta}\left(\|f^0_z\|^{q_0}_{L^{q_0}(B_0)} + z\|f^1_z\|^{q_1}_{L^{q_1}(B_1)}\right)\right]^{r/q}\frac{dz}{z}\\ &\geqq C'\|f\|^r_{(L^{q_0}(B_0),L^{q_1}(B_1))_{\theta,r}} \qquad (C'>0).\end{aligned}$$

Taking into account the embedding (5.2), we obtain (5.13) under the additional condition (5.14). The general case reduces to this one by using the reiteration theorem exactly as in the proof of Theorem 5.2. □

COROLLARY 5.7. *Let $B_i$ be of Paley $\Phi$-type* $p_i$ ($i=0,1$; $1<p_0$, $p_1<2$), $0<\theta<1$ and $1/p=(1-\theta)/p_0+\theta/p_1$. Then the space $(B_0,B_1)_{\theta,p'}$ is of Paley $\Phi$-type* $p$.*

THEOREM 5.8. *Let $\{B_0,B_1\}$ be an interpolation couple of Banach spaces such that $B_i$ is of strong Paley $\Phi$-cotype* $q_i$ ($i=0,1$; $2<q_1\leqq q_0<\infty$). Suppose that $0<\theta<1$, $1/q=(1-\theta)/q_0+\theta/q_1$ and $q'\leqq r\leqq q$. Then there exists a constant $A$ such that for every function $f\in L^{q',r}_B$ ($B\equiv B_{\theta,r}$).*

$$\left\|\left\{c_n(f)\right\}\right\|_{l^q_B} \leqq A\|f\|_{L^{q',r}_B}. \tag{5.21}$$

PROOF. It suffices to prove the theorem in the case when $f$ is a simple function. Moreover, since the space $(I,\mu)$ is non-atomic, we may assume that $f$ is a function of the form $f=\sum_{k=1}^{N} a_k\chi_{E_k}$, where $a_k\in B$ and $E_k\subset I$ are mutually disjoint measurable sets with $\mu(E_k)=\frac{1}{N}$ $(k=1,\dots,N)$.

If $r=q'$, then (5.21) follows from (5.1) and (5.4). Let $r>q'$. First we suppose that the condition (5.14) holds.

Let $\eta=\theta\frac{q}{q_1}$. Using the $L$-method, we choose for each $k=1,\dots,N$ the representation

$$a_k=\overline{a}_k^0(t)+\overline{a}_k^1(t)\quad \bigl(0<t<\infty,\ \overline{a}_k^i(t)\in B_i\bigr)$$

so that

(5.22)

$$\|a_k\|_B\geqq C_1\biggl(\int_0^\infty\Bigl[t^{-\eta}\bigl(\bigl\|\overline{a}_k^0(t)\bigr\|_{B_0}^{q_0}+t\bigl\|\overline{a}_k^1(t)\bigr\|_{B_1}^{q_1}\bigr)\Bigr]^{r/q}\frac{dt}{t}\biggr)^{1/r}\quad (C_1>0).$$

Furthermore, we have $(p\equiv q'<r)$

$$\|f\|_{L^{p,r}(B)}^r\geqq\frac{p}{r}N^{-r/p}\sum_{k=1}^{N}k^{r/p-1}(a_k^*)^r=\frac{p}{r}N^{-r/p}\sum_{k=1}^{N}\sigma_k^{r/p-1}\|a_k\|_B^r,$$

where $\{\sigma_k\}_{k=1}^N$ is some rearrangement of the indices $\{1,\dots,N\}$.

Let $a_k^i=\overline{a}_k^i(t\sigma_k^{-\gamma}N^{\gamma})$, $0<t<\infty$, $i=0,1$, where $\gamma=q_0-q_1$. Setting $s_i=q_ir/q$, we obtain (see the proof of Theorem 5.6)

$$\|f\|_{L^{p,r}(B)}^r\geqq C\int_0^\infty z^{-\eta r/q}\biggl[N^{-s_0/q_0'}\sum_{k=1}^{N}\sigma_k^{s_0/q_0'-1}\bigl\|a_k^0(z)\bigr\|_{B_0}^{s_0}$$

$$+z^{r/q}N^{-s_1/q_1'}\sum_{k=1}^{N}\sigma_k^{s_1/q_1'-1}\bigl\|a_k^1(z)\bigr\|_{B_1}^{s_1}\biggr]\frac{dz}{z}.$$

Let $f_z^i=\sum_{k=1}^N a_k^i(z)\chi_{E_k}$; then $f=f_z^0+f_z^1$ $(0<z<\infty)$. As $q_i'<s_i\leqq q_i$ (see (5.14)), we have

$$N^{-s_i/q_i'}\sum_{k=1}^{N}\sigma_k^{s_i/q_i'-1}\bigl\|a_k^i(z)\bigr\|_{B_i}^{s_i}\geqq\frac{q_i'}{s_i}\|f_z^i\|_{L^{q_i',s_i}(B_i)}^{s_i}\geqq C\|f_z^i\|_{L^{q_i',q_i}(B_i)}^{s_i}.$$

Thus, we obtain

$$\|f\|^r_{L^{p,r}(B)} \geqq C \int_0^\infty \left[ z^{-\eta} \left( \|f_z^0\|^{q_0}_{L^{q_0',q_0}(B_0)} + z \|f_z^1\|^{q_1}_{L^{q_1',q_1}(B_1)} \right) \right]^{r/q} \frac{dz}{z}.$$

Now we can complete the proof as in Theorem 5.6. $\square$

Corollary 5.9. *Let $B_i$ be of strong Paley $\Phi$-cotype* $q_i$ $(i = 0, 1;$ $2 < q_0, q_1 < \infty)$, $0 < \theta < 1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Then the space $(B_0, B_1)_{\theta,q}$ is of strong Paley $\Phi$-cotype* $q$.*

## §6 Estimates of coefficients for functions with values in Lorentz spaces

Let $(I, \mu)$ be a non-atomic measure space, $\mu(I) = 1$, and let $\Phi$ be a uniformly bounded orthonormal system on $I$. We shall consider series of functions belonging to the system $\Phi$ with coefficients from the Lorentz space $L^{p,r} \equiv L^{p,r}(\Omega, \nu)$, where $(\Omega, \nu)$ is a measure space. It is known that $L^{p,r}$ $(1 < p, r < \infty)$ is of Fourier type exactly $\min(p, p', r, r')$ (see [11] and [14]). This result was obtained by interpolation. Namely, we have for $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \leqq r < \infty$

$$L^{p,r} = (L^{p_0}, L^{p_1})_{\theta,r} \tag{6.1}$$

(see [20]).

Using Theorems 4.2, 5.2 and 5.4, we have the following result.

Theorem 6.1. *Let $B = L^{p,r}(\Omega, \nu)$ $(1 < p < 2,\ p \leqq r \leqq p')$ and $\Phi$ be an arbitrary uniformly bounded ONS on $I$. Then there exists a constant $A$ such that*

(1) *for every finite sequence $b = \{b_k\}_{k=1}^n$, $b_k \in B$,*

$$\|f\|_{L_B^{p',r}} \leqq A \|b\|_{l_B^p} \quad \text{where} \quad f = \sum_{k=1}^n b_k \varphi_k; \tag{6.2}$$

(2) *for every function $f \in L_B^p$*

$$\big\| c(f) \big\|_{l_B^{p',r}} \leqq A \|f\|_{L_B^p} \quad \text{where} \quad c(f) = \big\{ c_n(f) \big\}. \tag{6.3}$$

Corollary 6.2. *The space $L^{p,p'}$ $(1 < p < 2)$ is of Riesz $\Phi$-type $p$ and strong Riesz $\Phi$-cotype $p'$ for any uniformly bounded ONS $\Phi$.*

It was proved in [9] that the space $L^{p,r}$ $(1<p<2)$ is of Riesz $\Phi$-type $p$ and strong Riesz $\Phi$-cotype $p'$ for any uniformly bounded ONS $\Phi$ and any $p \leqq r \leqq p'$. If $p<r<p'$, then inequality (6.2) expresses a property which is weaker than being of Paley $\Phi$-type $p$ but stronger than being of Riesz $\Phi$-type $p$. The inequality (6.3) has a similar "intermediate" character. Of course, these inequalities are sharp with respect to $p$ (it follows already from the fact that $L^{p,r}$ is not of Fourier type greater than $p$ [11], [14]). We shall prove that inequalities (6.2) and (6.3) are also sharp in a stronger sense — with respect to the second Lorentz index.

Let $\Omega$ be the interval $[0,1]$ with Lebesgue measure.

PROPOSITION 6.3. *Let $B=L^{p,r}[0,1]$ $(1<p<2,\ p\leqq r\leqq p')$. The following assertions hold:*

(1) *Let $\Phi=\{\varphi_n\}$ be an arbitrary ONS on $[0,1]$ such that $\left|\varphi_n(x)\right|=1$ a.e. If $s>p$, then there exists a sequence $\{b_k\}\in l_B^{p,s}$ such that for almost all $x\in[0,1]$*

$$\Big\| \sum_{k=1}^{n} b_k\varphi_k(x)\Big\|_B=\lambda_n,$$

*where $\lim_{n\to\infty}\lambda_n=\infty$.*

(2) *If $s>p$ and $\sigma>1$, then there exists a sequence of trigonometric polynomials*

$$T_k(x)=\sum_{n=-n_k}^{n_k} b_n^{(k)}e^{2\pi inx} \tag{6.4}$$

*such that $\|T_k\|_{L_B^{p,s}}\leqq 1$ $(k=1,2,\dots)$, but*

$$\lim_{k\to\infty}\Big\|\{b_n^{(k)}\}_{n=-n_k}^{n_k}\Big\|_{l_B^{p',\sigma}}=\infty.$$

PROOF. (1) We use Lemma 4.3, setting

$$q=p,\quad a_k=(\ln k+1)^{1/p},\quad \eta_k=\frac{1}{\ln k+1}\quad (k\in\mathbf{N}).$$

Let $b_k(y)=a_k\chi_{\Delta_k}(y)$, $\Delta_k=(\eta_{k+1},\eta_k]$, $y\in[0,1]$, and

$$f_n(x)=\sum_{k=1}^{n} b_k\varphi_k(x).$$

The series (4.7)

$$\sum_{k=1}^{\infty} k^{-1}(\ln k+1)^{-s/p}$$

converges, while the series (4.8)

$$\sum_{k=1}^{\infty}\big(k(\ln k+1)\big)^{-1}$$

diverges. Next, we have

$$\big|f_n(x)(y)\big| = \sum_{k=1}^{n} a_k \chi_{\Delta_k}(y) \quad \text{for any} \quad x\in[0,1].$$

As in the proof of Proposition 4.4, we obtain that $\big\|f_n(x)\big\|_B = \lambda_n \to \infty$, for almost all $x$, while the sequence $\big\{\big\|\{b_k\}_{k=1}^n\big\|\big\}_{l_B^{p,s}}\}$ is bounded. This implies (1). Statement (2) follows from statement (1) and Lemma 3.1. $\square$

Thus, it is impossible to replace the norms on the right hand sides of (6.2) and (6.3) by weaker Lorentz norms, even if we simultaneously take weaker norms on the left hand sides.

Now we shall show that it is also impossible to improve (6.2) and (6.3) by means of strengthening the norms on the left hand side, even if at the same time we strengthen the norms on the right hand side.

PROPOSITION 6.4. *Let $B = L^{p,r}[0,1]$ $(1<p<2,\ p\leqq r\leqq p')$. Then:*

(1) *If $0<s<r$, then there exists a function $f\in L_B^{\infty}$ such that the partial sums $S_n(x)=\sum_{k=-n}^{n} c_k e^{2\pi i k x}$ have uniformly bounded $B$-norms but the sequence $\{c_k\}$ does not belong to the space $l_B^{p',s}$.*

(2) *If $0<s<r$ and $\sigma>1$, then there exists a sequence of trigonometric polynomials (6.4) such that*

$$\big\|\{b_n^{(k)}\}_{n=-n_k}^{n_k}\big\|_{l_B^{p,\sigma}} \leqq 1, \quad \textit{but} \quad \lim_{k\to\infty}\|T_k\|_{L_B^{p',s}}=\infty.$$

PROOF. (1) Let

$$g(z)=\sum_{n=1}^{\infty} a_n\cos 2\pi n z, \quad a_n = n^{-1/p'}\big(\ln(n+1)\big)^{-1/s}. \tag{6.5}$$

Define the function $f:[0,1]\to B$ by the relation

$$f(x)(y)=g(x+y), \quad x,y\in[0,1].$$

Its Fourier coefficients are

$$c_n(y)=\int_0^1 f(x)(y)e^{-2\pi i n x}\,dx=\frac{1}{2}a_n e^{2\pi i n y} \quad (n\geqq 1),$$

$$c_{-n} = \frac{1}{2} a_n e^{-2\pi i n y} \quad (n \geqq 1).$$

Thus, $\|c_n\|_B = \frac{1}{2}\left(\frac{p}{q}\right)^{1/r} a_{|n|}$. Since $\{a_n\} \notin l^{p',s}$, then $\{c_n\} \notin l_B^{p',s}$.

Next, for any fixed $x \in [0,1]$ the non-increasing rearrangement is

$$\big(f(x)\big)^*(t) = g^*(t), \quad 0 \leqq t \leqq 1.$$

But $g(z) \sim \big(z^{1/p} \ln^{1/s}(1/z)\big)^{-1}$, $z \to 0$ (see [19, p. 187]). Thus, $g \in L^{p,r}$, and for all $x \in [0,1]$

$$\big\| f(x)\big\|_{L^{p,r}} = \|g\|_{L^{p,r}}.$$

Let

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{2\pi i k x}, \quad \sigma_n(z) = \sum_{k=1}^{n} a_k \cos 2\pi k z.$$

Then

$$S_n(x)(y) = \sigma_n(x+y) \quad \text{and} \quad \big(S_n(x)\big)^*(t) = \sigma_n^*(t) \quad (0 \leqq t \leqq 1)$$

for any $x \in [0,1]$. But (see [21])

$$\big|\sigma_n(z)\big| \leqq \frac{C}{z^{1/p} \ln^{1/s}(1/z)} \quad \left(0 < z \leqq \frac{1}{2},\ n \in \mathbf{N}\right).$$

Thus, $\big\| S_n(x)\big\|_{L^{p,r}} = \|\sigma_n\|_{L^{p,r}} \leqq C'$ for any $x \in [0,1]$ and any $n \in \mathbf{N}$. This proves the statement (1). The statement (2) follows from (1) and Lemma 3.1. $\square$

Furthermore, by using Theorems 4.1, 5.6 and 5.8, and taking into account (6.1), we obtain the following result.

THEOREM 6.5. *Let $B = L^{q,r}(\Omega, \nu)$ $(2 < q < \infty,\ q' \leqq r \leqq q)$ and $\Phi$ be an arbitrary uniformly bounded ONS on $I$. Then there exists a constant $A$ such that*

(1) *for every finite sequence $b = \{b_k\}_{k=1}^n$, $b_k \in B$*

$$\|f\|_{L_B^q} \leqq A\|b\|_{l_B^{q',r}}, \quad \text{where} \quad f = \sum_{k=1}^{n} b_k \varphi_k; \tag{6.6}$$

(2) *for every function $f \in L_B^{q',r}$*

$$\big\| c(f)\big\|_{l_B^q} \leqq A\|f\|_{L_B^{q',r}}, \quad \text{where} \quad c(f) = \big\{c_n(f)\big\}. \tag{6.7}$$

COROLLARY 6.6. *The space* $L^{q,q'}$ $(2 < q < \infty)$ *is of Riesz* $\Phi$*-type* $q'$ *and strong Riesz* $\Phi$*-cotype* $q$ *for any uniformly bounded ONS* $\Phi$.

Theorem 6.5 is dual to Theorem 6.1; it is sharp in the same sense (see Propositions 6.3 and 6.4). We can use here Lemma 2.9. Indeed, if $(\Omega,\nu)$ is a non-atomic measure space, then the dual to $L^{p,r}(\Omega,\nu)$ may be identified with the space $L^{p',r'}(\Omega,\nu)$ (see [2, p. 221]).

Finally we consider the space $L^{p,r}$ for the case when $r$ does not lie between $p$ and $p'$. It is known (see [11] and [14]) that in this case $L^{p,r}$ is of Fourier type $\min(r,r')$ (and no better).

THEOREM 6.7. *Let* $B = L^{p,r}$ $(1 < p < \infty,\ p \neq 2,\ r > 1)$ *and* $\Phi$ *be an arbitrary uniformly bounded ONS on* $I$. *Then*

(1) *if* $r < \min(p,p')$, *then* $B$ *is of Paley* $\Phi$*-type* $r$ *and strong Paley* $\Phi$*-cotype* $r'$ *and*

(2) *if* $r > \max(p,p')$, *then* $B$ *is of Paley* $\Phi$*-type** $r'$ *and strong Paley* $\Phi$*-cotype** $r$.

PROOF. Let $1 < p < 2$, $r < p$ (in the other cases the proof is similar). Let us choose the numbers $p_0, p_1 \in (r,2)$ so that $p_0 < p < p_1$. Furthermore, define $\theta$ from the equation $1/p = (1-\theta)/p_0 + \theta/p_1$. Then $(L^{p_0}, L^{p_1})_{\theta,r} = L^{p,r}$ (see (6.1)). It follows from Theorems 4.2, 2.18 and 2.19 that the spaces $B_i = L^{p_i}$ $(i = 0,1)$ both are of Paley $\Phi$-type $r$ and strong Paley $\Phi$-cotype $r'$. It only remains to apply Corollaries 5.3 and 5.5. $\square$

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DEPARTAMENTO DE MATEMÁTICAS
C-XV, UNIVERSIDAD AUTÓNOMA
28049, MADRID
SPAIN
E-MAIL: JOSE.GARCIA-CUERVA@UAM.ES
KAZAROS.KAZARIAN@UAM.ES

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN
EDIF.VIVES, C. LUIS DE ULLOA
UNIVERSIDAD DE LA RIOJA
26004 LOGROÑO, LA RIOJA
SPAIN
E-MAIL: VICTOR.KOLYADA@DMC.UNIRRIOJA.ES