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A simple method for the synthesis of 2D and 3D mechanisms with kinematic constraints

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Abstract

A general procedure for the synthesis of mechanisms is presented here. This approach minimises the error between the actual path of one or several points of the mechanism and the paths for each of them predefined by a certain number of points. It is also possible to consider kinematic constraints on velocity, acceleration and jerk. The optimisation method uses a sequence of quadratic problems with an analytical definition of the objective function, constraints and their gradients, while the hessians are computed by finite differences.

Keywords: Synthesis of mechanisms; Topological and kinematic constraints; 2D and 3D path optimisation

1. Introduction

During the last years, there have been important advances in the problem of optimal synthesis of mechanisms, mainly due to the exponential development of computer performances, together with successive improvements in optimisation methodologies [2–7]. This has allowed the application of different mathematical programming techniques to the dimensional and topological synthesis of mechanisms [4,8].

However, most of the available procedures have been developed for particular problems, usually hiding fundamental physical aspects that are relevant for the designer, and, in general, leading to methods difficult to apply in the day by day engineering practice. The method

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proposed in this paper tries to solve some of these disadvantages, including in the scheme most of the usual situations in mechanism design, such as:

- The possibility of solving 2D and 3D open or closed chain mechanisms under a unique systematics.
- The ability of considering topological, dimensional and location constraints.
- The inclusion of lower and higher order pairs without any special distinction between them.
- The treatment of kinematic constrains on velocity, acceleration and jerk of any point and along any direction, using an identical scheme than the one used for the rest of the constraints.
- The use of the so-called *natural co-ordinates* as in García de Jalón et al. [5,6] which allows an easier treatment of the constraints and a straightforward geometrical representation of the movement of the mechanism.

With all of this, a very general and close to standard practice method has been obtained with a reliable behaviour even when the initial design is not very close to the optimal one.

The analysis procedure, following previous works of Avilés et al. [3], Navalpotro [9] and Vallejo [12], is divided into two stages, the so-called *Local and Global Synthesis*.

The objective of the first stage is to obtain an initial iteration vector good enough to start with guarantee the actual optimisation process. An auxiliary optimisation problem is solved for each synthesis point in order to get the position of the mechanism closest to the proposed synthesis point, using as design variables the natural co-ordinates of the mechanism, fixing the element lengths.

On the contrary, in the second stage, the actual optimisation problem is solved, adding the dimensions of the mechanism as new design variables. With this, the best mechanism in the sense of the chosen objective function fulfilling the topological and kinematic constraints is obtained.

The *objective function* is the same for both stages and corresponds to the one previously used in [9,12]. It is defined as the *strain energy* of the bars of the mechanism,[1] considered as flexible elements, needed to reach, from a certain position of the mechanism compatible with its actual lengths, a certain synthesis point. For the local synthesis stage, the position of the mechanism, which needs less strain energy to reach the synthesis point, is therefore obtained. The vector formed by the addition of all the local solutions of these "deformed" mechanisms, that is, the positions which actually pass through each of the synthesis points are considered as the initial iteration vector for the Global Synthesis stage. This uses the same objective function but now written for all the synthesis points at the same time.

There are some interesting properties of this objective function which makes it useful for this kind of problems. First of all its scalar character. In second place, its physical meaning that allows the trained designer to discuss the results from a physical point of view much closer to his knowledge and interest. Finally, the possibility of including weighting parameters in a very simple and physical way (i.e. by an appropriate choice of the elastic parameters of the materials that compose each bar). This last makes it possible to weight the relative importance

[1] In fact, any dimensional constraint established in terms of length is considered a bar for this purpose.

of the change of length of each element during the design process (i.e. a "stiffer" bar would mean a more important addition to the global strain energy with respect to more "flexible" bars, and therefore, a lower probability of change of its length).

With respect to constraints, besides the typical topological ones, there exists additionally the possibility of considering constraints on velocity, acceleration or jerk. It is also interesting to point out that it is possible to include user constraints by defining the corresponding function and Jacobian with respect to the design variables by the appropriate user routines. All of the constraints are defined by a design region established by appropriate lower and upper bounds related to the required accuracy of the solution. This scheme also allows to relax the fulfilment of the constraints according to the tolerance admitted by the designer, working therefore with *synthesis regions* instead of synthesis points allowing a greater adaptation of the mechanism to the proposed design, specially when kinematic constraints are present. This is also closer to the actual design process where a certain tolerance for each point is allowed in order to get a better solution.

The optimisation algorithm that has been used is SQP (sequence of quadratic problems) [7,13], a robust algorithm with a good convergence ratio, although in this work the former feature has been considered preferable to the latter. To solve it the subroutine E04VDF of the commercial mathematical library NAG [11] has been used. This routine employs the analytical definition of the objective function, the constraints and their Jacobians; while the hessians are computed by a finite difference approach. The explanation of the whole process, including the formulation of the objective and constraint functions is presented in the following paragraphs for a problem without kinematic constraints, which are added in Section 3. Finally, in Section 4, a complete set of examples including different types of situations are analysed in order to show the global performance of the method, its reliability and possibilities, finishing in Section 5 with several conclusions and recommendations.

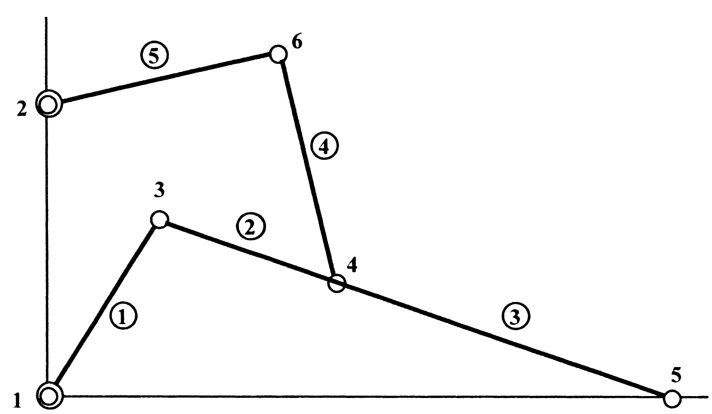

Fig. 1. Model of a mechanism.

2. Synthesis of mechanisms with topological constraints

2.1. Design variables

A mechanism is defined in this work by a set of elements (bars) and couples (pairs) which compose its topology (see Fig. 1). The co-ordinates used to define the kinematics (and dynamics) of the mechanism are the so-called natural co-ordinates of García de Jalón et al. [5,6]. This type of co-ordinates are based on the standard degrees of freedom used in the well-known Finite Element Method [14] in structural analysis, appearing similar concepts like nodes (points to which the degrees of freedom are attached) and elements (bars of the mechanism). The Cartesian co-ordinates of the nodes define the position of the mechanism, while the displacements of these nodes establish its location along time. In general (this is specially true in 3D problems), additional degrees of freedom have to be employed (i.e. rotations).

Naturally, the definition of a mechanism also includes some constraints on the variation of the co-ordinates (degrees of freedom) induced by elements and couples. For example, a bar between two nodes implies a fixed distance (length of the bar) between those points along the movement, that is a constraint between the degrees of freedom attached to those two nodes; a fixed angle between two elements induced by a prismatic couple; the movement of a node along a certain direction induced by a slider or a fixed co-ordinate of a fixed point. Besides the natural co-ordinates, only the bar lengths have been considered here as design variables, being straightforward to generalise this formulation to include additional design variables. Of course, all the above are only potential design variables, since some of them may be fixed parameters of the problem. Each design variable is defined in the input file by its range of variation (a lower and an upper bound) which defines its feasible design region.

It is interesting to point out that the element lengths are design variables only for the actual optimisation process that is the global synthesis stage, while the natural co-ordinates are design variables both for the local and global synthesis approaches. Summarising, the design variables of the problem are:

- *Local synthesis*: Natural co-ordinates of the mechanism I associated to the synthesis point I (usually the co-ordinates of the nodes) except the ones corresponding to fixed points.
- *Global synthesis*: Natural co-ordinates of the complete set of I-mechanisms, plus the co-ordinates of the fixed points, plus the lengths of the elements of the mechanism.

The inclusion of all the natural co-ordinates in the set of design variables, although expensive in time, allows a very easy treatment of any constraint, including kinematic ones. In fact, in this work, simplicity and reliability have been considered as primary conditions with respect to computer cost (which is a rapidly decreasing condition in computational mechanics).

2.2. Constraints

The next step is the definition of the constraints on the design variables in order, not only to define the global topology of the mechanism, but to impose the actual design constraints on the design variables. For planar mechanisms, the main constraints considered have been:

- fixed points constraints, which impose the constancy of the co-ordinates of the supports for

each synthesis position
- a three-bar triangle constraint defining a rigid body condition
- co-linearity of three points
- constant angle between two bars
- fixed direction of the movement of one point
- gear–gear pair
- crank–gear pair

for spatial mechanisms, the main implemented constraints are:

- rigid triangle
- rigid tetrahedron
- spatial co-linearity
- co-planarity
- constant angle between two bars
- helical joint
- cylindrical joint

Finally, any other constraint that can be expressed analytically can be included with no more than writing its analytical expression and the corresponding gradient with respect to each of the design variables. This is needed to formulate the Jacobian in the SQP algorithm used to solve the optimisation problem.

In the next paragraphs, as useful examples, some of these constraints and their derivatives are formulated in terms of the design variables. A more detailed explanation of the different constraints and their corresponding analytical treatment may be found in Ref. [1].

2.2.1. Fixed points

Co-ordinates of fixed points may be considered as parameters of the mechanism, and therefore, not included in the design variable vector. However, in some problems, the fixed-point co-ordinates may be design variables moving between certain bounds. In these cases, the values of these variables remain unchanged for each location of the mechanism and, in particular, for the synthesis positions. These constraints are written, therefore, in the form:

$$\begin{aligned} &h(j,i,k,l) = x_j\big[FP_l(i)\big] - x_j\big[FP_l(k)\big] = 0 \\ &\quad j = 1,2,3;\ i = 1,\dots,SP;\ k = 1,\dots,SP;\ l = 1,\dots,FP \end{aligned} \tag{1}$$

with SP the number of synthesis points and FP the number of fixed points.

The Jacobian of these constraints is just

$$\begin{aligned} &\frac{\partial h(j,i,k,l)}{\partial x_m\big[FP_n(o)\big]} = \delta_{jm}\delta_{ln}(\delta_{io} - \delta_{ko}) \\ &\quad j = 1,2,3;\ i = 1,\dots,SP;\ k = 1,\dots,SP;\ l = 1,\dots,FP \end{aligned} \tag{2}$$

with δ_{ij} the Kronecker symbol.

2.2.2. Point moving along a certain direction (Fig. 2)

Its analytical expression is written as

$$h_1\left(x_A, y_A, x_A^P, y_A^P, v_1, v_2\right) \equiv \frac{x_A - x_A^P}{v_1} - \frac{y_A - y_A^P}{v_2} = 0$$

$$h_2\left(x_A, z_A, x_A^P, z_A^P, v_1, v_3\right) \equiv \frac{x_A - x_A^P}{v_1} - \frac{z_A - z_A^P}{v_3} = 0 \tag{3}$$

where $\boldsymbol{x}_A^P$ denotes the location of a fixed point along the fixed direction $\boldsymbol{v}$.

The Jacobian is immediately

$$\frac{\partial h_j}{\partial x_A} = \frac{1}{v_1}\frac{\partial h_j}{\partial y_A} = -\frac{\delta_{j1}}{v_2}\frac{\partial h_j}{\partial x_A} = -\frac{\delta_{j2}}{v_3} \qquad \frac{\partial h_j}{\partial x_A^P} = \frac{1}{v_1}\frac{\partial h_j}{\partial y_A^P} = -\frac{\delta_{j1}}{v_2}\frac{\partial h_j}{\partial x_A^P} = -\frac{\delta_{j2}}{v_3}$$

$$\frac{\partial h_j}{\partial v_1} = -\frac{x_A - x_A^P}{v_1^2} \qquad \frac{\partial h_j}{\partial v_2} = -\delta_{j1}\frac{y_A - y_A^P}{v_2^2} \qquad \frac{\partial h_j}{\partial v_3} = -\delta_{j2}\frac{z_A - z_A^P}{v_3^2} \tag{4}$$

2.2.3. Constant sign of the cosine or sine of the angle between two bars

This constraint appears when controlling the relative position of three points defining a rigid body in order to keep one of the points inside one of the half-planes defined by the line joining the other two. It is written as a condition on the dot product between the bar vectors

$$1 \geq h(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C) = \frac{\boldsymbol{r}_{BA} \bullet \boldsymbol{r}_{CA}}{L_{AB}L_{AC}} \geq 0 \tag{5a}$$

for a positive cosine.

In terms of the design variables

$$h(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C) = \frac{x_{BA}x_{CA} + y_{BA}y_{CA} + z_{BA}z_{CA}}{L_{AB}L_{AC}} \tag{5b}$$

with

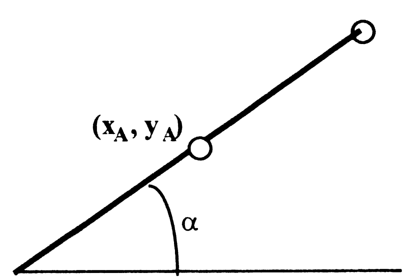

Fig. 2. Point moving along a certain direction.

$$x_{BA} = x_B - x_A \qquad y_{BA} = y_B - y_A \qquad z_{BA} = z_B - z_A$$

$$x_{CA} = x_C - x_A \qquad y_{CA} = y_C - y_A \qquad z_{CA} = z_C - z_A$$

$$L_{AB} = \sqrt{x_{AB}^2 + y_{AB}^2 + z_{AB}^2} \qquad L_{AC} = \sqrt{x_{AC}^2 + y_{AC}^2 + z_{AC}^2} \tag{6}$$

The corresponding components of the Jacobian may be written as

$$\frac{\partial h}{\partial x_A^j} = \frac{\left(-x_{CA}^j - x_{BA}^j\right)(L_{AB}L_{AC}) - \boldsymbol{r}_{BA} \bullet \boldsymbol{r}_{CA}\left[\frac{\partial L_{AB}}{\partial x_A^j}L_{AC} + L_{AB}\frac{\partial L_{AC}}{\partial x_A^j}\right]}{L_{AB}^2 L_{AC}^2}$$

$$\frac{\partial h}{\partial x_B^j} = \frac{x_{CA}^j(L_{AB}L_{AC}) - \boldsymbol{r}_{BA} \bullet \boldsymbol{r}_{CA}\frac{\partial L_{AB}}{\partial x_B^j}L_{AC}}{L_{AB}^2 L_{AC}^2}$$

$$\frac{\partial h}{\partial x_C^j} = \frac{x_{BA}^j(L_{AB}L_{AC}) - \boldsymbol{r}_{BA} \bullet \boldsymbol{r}_{CA}\frac{\partial L_{AC}}{\partial x_C^j}L_{AB}}{L_{AB}^2 L_{AC}^2} \tag{7}$$

With respect to the constant sign of a sine, it appears in cases when not only a direction but a certain sense has to be kept. It is imposed as a cross product condition, that is

$$1 \geq h(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C) \equiv \frac{\boldsymbol{r}_{BA} \times \boldsymbol{r}_{CA}}{L_{AB}L_{AC}} \geq 0 \tag{8}$$

The gradient is given by similar expressions to Eq. (7).

2.2.4. Triangle constraint

Besides the standard bar length constraints, it is necessary to ensure that the triangle does not turn around any of its sides. This is established by imposing the constancy of the sign of the sine and cosine of two of the angles of the triangle. For example, on making reference to

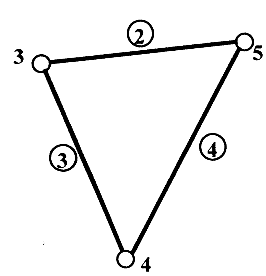

Fig. 3. Triangle constraint.

Fig. 3, we have

$$\sin(\alpha) = \frac{\boldsymbol{r}_{35} \times \boldsymbol{r}_{34}}{L_2 L_3} > 0 \qquad \cos(\alpha) = \frac{\boldsymbol{r}_{35} \bullet \boldsymbol{r}_{34}}{L_2 L_3} > 0 \qquad \cos(\beta) = \frac{\boldsymbol{r}_{35} \bullet \boldsymbol{r}_{45}}{L_2 L_4} > 0 \tag{9}$$

with immediate gradient.

2.2.5. Gear–gear pair

This situation is shown in Fig. 4. The mathematical constraint establishes that the length of the arc followed by point A_1 from its initial position $(A_1)_0$ has to be the same that the one followed by point A_2 from $(A_2)_0$

$$h_5 = \left[\theta_{A1} - (\theta_{A1})_0\right]L_1 + \left[\theta_{A2} - (\theta_{A2})_0\right]L_2 = 0 \tag{10a}$$

with

$$\theta_{A1} = \text{arctg}\frac{y_{A1} - y_{01}}{x_{A1} - x_{01}} \qquad \theta_{A2} = \text{arctg}\frac{y_{A2} - y_{02}}{x_{A2} - x_{02}}$$

$$L_1 = \sqrt{(x_{A1} - x_{01})^2 + (y_{A1} - y_{01})^2} \qquad L_2 = \sqrt{(x_{A2} - x_{02})^2 + (y_{A2} - y_{02})^2} \tag{10b}$$

with gradients

$$\frac{\partial h_5}{\partial x_{A1}} = \frac{\partial \theta_{A1}}{\partial x_{A1}}L_1 + \left[\theta_{A1} - (\theta_{A1})_0\right]\frac{\partial L_1}{\partial x_{A1}} \qquad \frac{\partial h_5}{\partial y_{A1}} = \frac{\partial \theta_{A1}}{\partial y_{A1}}L_1 + \left[\theta_{A1} - (\theta_{A1})_0\right]\frac{\partial L_1}{\partial y_{A1}}$$

$$\frac{\partial h_5}{\partial x_{01}} = \frac{\partial \theta_{A1}}{\partial x_{01}}L_1 + \left[\theta_{A1} - (\theta_{A1})_0\right]\frac{\partial L_1}{\partial x_{01}} \qquad \frac{\partial h_5}{\partial y_{01}} = \frac{\partial \theta_{A1}}{\partial y_{01}}L_1 + \left[\theta_{A1} - (\theta_{A1})_0\right]\frac{\partial L_1}{\partial y_{01}}$$

$$\frac{\partial h_5}{\partial x_{A2}} = \frac{\partial \theta_{A2}}{\partial x_{A2}}L_2 + \left[\theta_{A2} - (\theta_{A2})_0\right]\frac{\partial L_2}{\partial x_{A2}} \qquad \frac{\partial h_5}{\partial y_{A2}} = \frac{\partial \theta_{A2}}{\partial y_{A2}}L_2 + \left[\theta_{A2} - (\theta_{A2})_0\right]\frac{\partial L_2}{\partial y_{A2}}$$

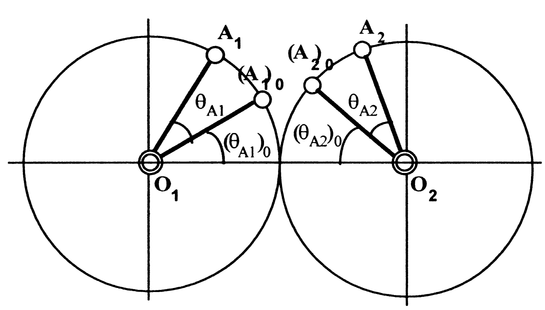

Fig. 4. Gear–gear constraint.

$$\frac{\partial h_5}{\partial x_{02}} = \frac{\partial \theta_{A2}}{\partial x_{02}} L_2 + \left[\theta_{A2} - (\theta_{A2})_0\right] \frac{\partial L_2}{\partial x_{02}} \qquad \frac{\partial h_5}{\partial y_{02}} = \frac{\partial \theta_{A2}}{\partial y_{02}} L_2 + \left[\theta_{A2} - (\theta_{A2})_0\right] \frac{\partial L_2}{\partial y_{02}} \tag{11}$$

and

$$\frac{\partial L_1}{\partial x_{A1}} = \frac{(x_{A1} - x_{01})}{\sqrt{\left[(x_{A1} - x_{01})^2 + (y_{A1} - y_{01})^2\right]}}$$

$$\frac{\partial L_1}{\partial y_{A1}} = \frac{(y_{A1} - y_{01})}{\sqrt{\left[(x_{A1} - x_{01})^2 + (y_{A1} - y_{01})^2\right]}}$$

$$\frac{\partial L_1}{\partial x_{01}} = -\frac{(x_{A1} - x_{01})}{\sqrt{\left[(x_{A1} - x_{01})^2 + (y_{A1} - y_{01})^2\right]}}$$

$$\frac{\partial L_1}{\partial y_{01}} = -\frac{(y_{A1} - y_{01})}{\sqrt{\left[(x_{A1} - x_{01})^2 + (y_{A1} - y_{01})^2\right]}}$$

$$\frac{\partial \theta_{A1}}{\partial x_{A1}} = \frac{1}{1 + \left(\frac{y_{A1} - y_{01}}{x_{A1} - x_{01}}\right)^2} \frac{y_{01} - y_{A1}}{(x_{A1} - x_{01})^2} \qquad \frac{\partial \theta_{A1}}{\partial y_{A1}} = \frac{1}{1 + \left(\frac{y_{A1} - y_{01}}{x_{A1} - x_{01}}\right)^2} \frac{1}{(x_{A1} - x_{01})}$$

$$\frac{\partial \theta_{A1}}{\partial x_{01}} = \frac{1}{1 + \left(\frac{y_{A1} - y_{01}}{x_{A1} - x_{01}}\right)^2} \frac{y_{A1} - y_{01}}{(x_{A1} - x_{01})^2} \qquad \frac{\partial \theta_{A1}}{\partial y_{01}} = \frac{1}{1 + \left(\frac{y_{A1} - y_{01}}{x_{A1} - x_{01}}\right)^2} \frac{-1}{(x_{A1} - x_{01})} \tag{12}$$

It has to be remarked that Eqs. (10a) and (10b) do not prevent the separation between gears that have to be kept in contact. This implies the necessity of imposing an additional constraint establishing that the distance between the centres of the gears has to be equal to the addition of the lengths of the two gear bars. This constraint may be written as

$$L_1 + L_2 = d_{01-02} \tag{13}$$

with

$$L_1 = \sqrt{(x_{A1} - x_{01})^2 + (y_{A1} - y_{01})^2} \qquad L_2 = \sqrt{(x_{A2} - x_{02})^2 + (y_{A2} - y_{02})^2}$$

$$d_{01-02} = \sqrt{(x_{02} - x_{01})^2 + (y_{02} - y_{01})^2}$$

and immediate gradients from Eq. (12).

2.2.6. Crank–gear pair

This case is similar to the previous one, except that for the length of the arc followed by a certain point A of the gear, from the initial contact point with the crank, has to be equal to the displacement of the corresponding point of the crank (Fig. 5).

In a similar way to the equation written for the gear–gear constraint, we have now

$$h_6 = \left[\theta_{A1} - (\theta_{A1})_0\right]L_1 - \sqrt{\left[x_{A2} - (x_{A2})_0\right]^2 + \left[y_{A2} - (y_{A2})_0\right]} = 0 \tag{14}$$

with

$$\theta_{A1} = \operatorname{arctg}\frac{y_{A1} - y_{01}}{x_{A1} - x_{01}}$$

The gradients with respect to the gear design variables are identical to the ones established in Eq. (11), while the ones for the crank variables are straightforward.

In the same way that was explained before for the gear–gear pair, an additional constraint has to be included in order to keep in contact the crank with the gear. This new constraint is written as

$$L_1^2 = (y_{01} - y_{A1})^2 \qquad L_1 = \sqrt{(x_{A1} - x_{01})^2 + (y_{A1} - y_{01})^2} \tag{15}$$

meaning that the distance between the centre of the gear and the crank has to be equal to the radius of the gear. The gradient of this constraint is again immediate from Eq. (12).

2.3. Objective function

The objective function, as was pointed out in Section 1, is defined for the local synthesis problem as

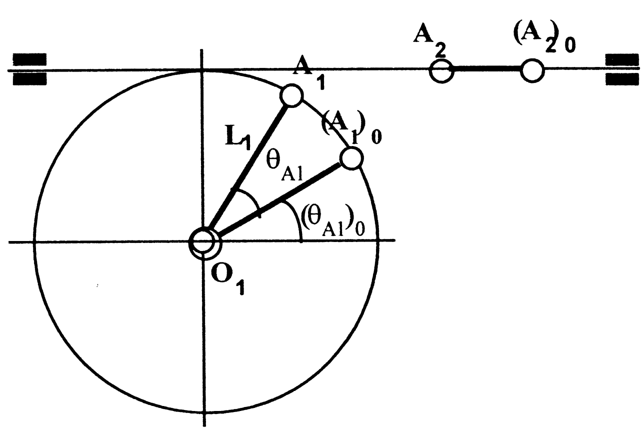

Fig. 5. Crank–gear constraint.

$$V = \sum_{i=1}^{m} E_i (l_i - L_i)^2 \tag{16}$$

with E_i the fictitious elastic module of each bar i, l_i the final length of the bar i to achieve the synthesis point, L_i the initial length of the bar i and m the number of bars of the mechanism able to be "deformed" (unfixed length) along the optimisation problem.

It has to be noticed that the initial iteration vector for the nodal co-ordinates has to be chosen as close as possible (in the sense of the "strain energy") to the final expected position of the mechanism, for each of the synthesis points. In order to do that, an initial analysis of the behaviour of the mechanism along a sufficient time interval is very useful, being possible to use a specific analysis program or even the local synthesis software. In this latter case, additional constraints establishing the value of the basic degrees of freedom along the movement have to be defined, while the synthesis constraints are discarded.

Fixed points are not considered as design variables in the local synthesis problem and kinematic constraints are not active in this case either.

The associated solutions give the co-ordinates of all the nodes of the mechanism for each synthesis point, that is for each associated "active" mechanism, being obvious that this solution depends on the initial lengths and the initial position. However, and as has been repeatedly obtained for the different examples studied, except for initial positions very far away from the optimal ones, the final result is almost independent of these initial values. On the contrary, this initial iteration vector affects dramatically the convergence ratio, especially when it is not close enough to the optimal solution.

The true optimisation problem is solved in the second stage (the so-called "global synthesis problem"). In this second procedure, the design variable vector is of dimension $\dim(\boldsymbol{x}) = 2 \times \text{NODES} \times \text{SP} + \text{BARS}$, which corresponds to each nodal co-ordinate for each synthesis point plus the bar lengths.

The objective function is again Eq. (16) but now extended to all the synthesis points, including additionally the bar lengths as new design variables and possible weighting coefficients w_k for each synthesis point k, that is

$$V = \sum_{k=1}^{PS} w_k \sum_{i=1}^{m} E_i (l_i - L_i)^2 \tag{17}$$

All the topological constraints considered in the local synthesis problem are again included here, expanding them to take into account all the different synthesis points, that is, writing them for each "active" mechanism of the problem.

Due to the fact that the design vector includes all the nodal co-ordinates of the mechanism for each synthesis position, we have, implicitly, the whole information needed to define completely the behaviour of the mechanism at each synthesis point in terms of the design variables, including its kinematic performance (velocities and accelerations of any point of the mechanism).

That is, for each synthesis point, an active mechanism I which fulfils the local synthesis but whose lengths are not, in general, the ones of the optimal solution of the problem is obtained. For each of these active mechanisms the analytical relations between their associated kinematic variables (velocities, accelerations and jerks) and the design variables are known, being possible to define additional constraints on kinematic values by the corresponding equivalent constraints among the design variables. This allows the control of velocities, accelerations, etc., during the design stage.

3. Synthesis with kinematic constraints

As it has been previously pointed out, for each synthesis point an *active mechanism I* is generated which fulfils the topological constraints but whose dimensions are not coincident, in general, with the ones of the final optimised mechanism. The velocity and acceleration of any point of the mechanism, at the time associated to any of the proposed active mechanisms, can be obtained in terms of the natural co-ordinates of this specific mechanism and the bar lengths, that is, in terms of the proposed design variables. Therefore, they may be considered as an implicit function of the design variables of the problem.

With this in mind, it is clear that the establishment of any kind of kinematic constraint on velocity, acceleration, jerk or any possible combination, may be considered as a new constraint on the design variables. The only condition is that these kinematic values correspond to the times of the synthesis points, which define the design variables.[2] This allows to control the values of these kinematic variables during the design process by adding these new constraints to the topological ones. However, these tend to constrain strongly the optimal solution leading, in some occasions, to an impossible one, becoming again critical in the initial analysis of the mechanism.

3.1. Formulation of the kinematic problem

The velocity problem is formulated by no more than deriving the initial position problem with respect to time, becoming a simple linear problem in terms of velocities at a certain predefined location of the mechanism.

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \dot{\boldsymbol{b}} \tag{18}$$

with $\dot{\boldsymbol{x}}$ the velocity vector (derivatives with respect to time of the natural co-ordinates of the mechanism), $\boldsymbol{E}$ the gradient matrix of the constraints with respect to the natural co-ordinates and $\dot{\boldsymbol{b}}$ the time derivative of the right-hand side vector of the constraints. Inverting Eq. (18) velocities are computed. It has to be noticed that $\boldsymbol{E}$ and $\dot{\boldsymbol{b}}$ are computed at a certain location, which corresponds to a certain time that, as was stated, is associated to one of the proposed synthesis points. To formulate Eq. (18), the time evolution of the drivers has to be known in order to compute the vector $\dot{\boldsymbol{b}}$.

When deriving the constraints with respect to time, we have to take into account that:

[2] Notice that constraints combining values of kinematic variables at different times may also be considered if all of them are associated to active mechanisms.

- the derivatives with respect to time of the bar lengths vanish, since they do not change along time.
- the derivative with respect to time of the fixed co-ordinates is obviously zero.
- for each synthesis point, velocities have to be analytic functions of the design variables, and therefore, derivable.

With this, it is possible to obtain immediately the gradients of the velocities with respect to the design variables in order to be included in the optimisation algorithm. This is accomplished by deriving Eq. (18) with respect to each of the design variables having

$$\boldsymbol{E}\left\{\frac{\partial \dot{\boldsymbol{x}}}{\partial x_j(i)}\right\} = \left\{\frac{\partial \dot{\boldsymbol{b}}}{\partial x_j(i)}\right\} - \left[\frac{\partial \boldsymbol{E}}{\partial x_j(i)}\right]\dot{\boldsymbol{x}} \quad j = 1, 2, 3;\ i = 1, \ldots, NODES \tag{19}$$

The gradient of each velocity component with respect to the design variables is, therefore, computed by inverting Eq. (19), taking into account that the system matrix $\boldsymbol{E}$ is the same that has been already inverted in Eq. (18). Therefore, the computer time needed is only the one associated to an additional back substitution of a new right-hand side. This is done for each of the synthesis points for which a kinematic constraint is established.

The acceleration problem is obtained in a similar manner by now deriving Eq. (18) with respect to time. With this, we have

$$\boldsymbol{E}\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{b}} - \dot{\boldsymbol{E}}\dot{\boldsymbol{x}} \tag{20}$$

This is computed after velocities, appearing again the same matrix $\boldsymbol{E}$ previously factorised.

In the same way, the gradients of the accelerations with respect to the design variables are computed by deriving Eq. (20) having

$$\boldsymbol{E}\left\{\frac{\partial \ddot{\boldsymbol{x}}}{\partial x_j(i)}\right\} = \left\{\frac{\partial \ddot{\boldsymbol{b}}}{\partial x_j(i)}\right\} - \left[\frac{\partial \dot{\boldsymbol{E}}}{\partial x_j(i)}\right]\dot{\boldsymbol{x}} - \dot{\boldsymbol{E}}\left\{\frac{\partial \dot{\boldsymbol{x}}}{\partial x_j(i)}\right\} - \left[\frac{\partial \boldsymbol{E}}{\partial x_j(i)}\right]\ddot{\boldsymbol{x}} \tag{21}$$
$$j = 1, 2, 3;\ i = 1, \ldots, NODES$$

Of course, and in the same manner that was discussed for velocities, Eqs. (20) and (21) are written for any of the active mechanisms where constraints on accelerations are established.

This allows us to include in the optimisation problem the desired kinematic constraints both on the values of their three components directly or on any combination of them like, for instance, the modulus

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{22}$$

with gradient

$$\frac{\partial v}{\partial x_i} = \frac{1}{v}\left(\dot{x}\frac{\partial \dot{x}}{\partial x_i} + \dot{y}\frac{\partial \dot{y}}{\partial x_i} + \dot{z}\frac{\partial \dot{z}}{\partial x_i}\right) \tag{23}$$

4. Examples

In the following paragraphs some examples are presented related to both the position and kinematic synthesis processes. They try to be representative of the possibilities of the method, including its capabilities and some of the drawbacks and problems already stated which usually imply a deep study of the performance of the mechanism previous to the proper synthesis problem.

4.1. Example 1: five bar mechanism

This is a very simple problem with exact and known solution, and therefore, its interest relies mainly on the study of the accuracy and convergence rate of the method in different situations. It corresponds to the mechanism shown in Fig. 6, with five nodes, five bars with lengths and connectivity included in Table 1, and two fixed points, nodes 1 and 2.

The topology constraints, and besides the constant bar lengths and fixed points co-ordinates, are in this case only the condition needed to keep the triangle topology, that can be expressed by the following three constraints

$$-1.00 < \sin(34 - 35) < 0.00 \qquad 0.00 < \cos(34 - 35) < 1.00 \qquad 0.00 < \cos(43 - 45) < 1.00$$

We shall study here the different solutions obtained when starting from different lengths of the bars imposing the synthesis point co-ordinates for node 5 included in Table 2, which correspond to exact co-ordinates of the analytical solution mechanism. The obtained results are shown in Table 3 for a fixed small number of iterations. From them, it is obvious that the initial choices 6, 7 and 14 are able to obtain the exact solution in that number of iterations, while solution 13 is close but could not. The rest of alternatives lead to results far from the actual solution. This approach is very simple, not very costly and allows the designer, when he has not an accurate idea of the expected solution, to obtain a proper initial iteration vector for the bar lengths.

If we now choose the lengths corresponding to case 13 as starting vector, and increase the accuracy not limiting the number of iterations, we get as final solution the exact mechanism with an evolution of the objective function shown in Fig. 7. Also, Fig. 8 shows, as an example, the following graphs for the synthesis point number 3:

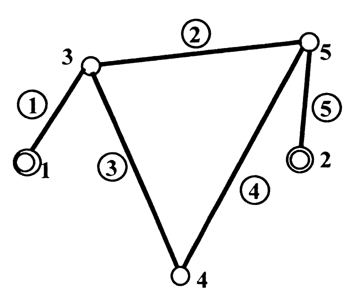

Fig. 6. Five bars mechanism.

Table 1
Topology of the mechanism

Bar	Node 1	Node 2	Length
1	1	3	4.000000
2	3	4	14.000000
3	3	5	15.000000
4	4	5	15.000000
5	2	4	6.000000

- Initial position of the active mechanism number 13 (grey line).
- Closest position of the initial mechanism to point 3 (local synthesis) (black thin line).
- Final position of the mechanism after the synthesis process (black thick line).

After this synthesis of position we move further imposing kinematic constraints. Of course, a previous analysis of the mechanism is very important in order to have an accurate idea of the range of velocities and accelerations of the different points of the mechanism. In this sense, for instance, the velocity of point 5 along direction x and at the time corresponding to the synthesis point 3 is $\dot{x}(5)_3 = -0.462977745\mathrm{D}+03$. This is the quantity we shall now restrain. Five different cases, as it is detailed in Table 4, have been considered as possible synthesis situations:

Case 1: A very strong constraint on this velocity, limiting it to $\dot{x}(5)_3 = -0.10000000\mathrm{D}+03$.
Case 2: A weak constraint onto the velocity, limiting it to $\dot{x}(5)_3 = -0.40000000\mathrm{D}+03$.
Case 3: The same constraint as in Case 2 plus a constraint on the y component of the acceleration of the same point at the same time. If the exact value for this quantity is $\dot{y}(5)_3 = -0.29895105\mathrm{D}+05$, we restrain its value to $\dot{y}(5)_3 = -0.25000000\mathrm{D}+05$.
Case 4: The same as in Case 3 but now the fixed points are allowed to move into a narrow band. This will allow a much easier adaptation of the mechanism to the imposed conditions, and therefore, a closer solution to the one searched (average of all the different constraints) shown in a substantial reduction of the objective function. The following bands have been allowed for the co-ordinates of the fixed points (NODE 1 = (0.0, 0.0); NODE 2 = (15.0, −1.0))

Table 2
Synthesis points

SP	*x*	*y*
1	13.265770	−10.019700
2	9.628700	−8.930700
3	7.757700	−8.838100
4	4.283400	−11.171100
5	6.413600	−12.612500
6	11.026300	−11.900400
7	14.978300	−10.341000

Table 3
Pseudo random starting vectors for the bar lengths

Case	Bar 1	Bar 2	Bar 3	Bar 4	Bar 5	Objective function
Initial lengths and final value of the objective function after five iterations						
1	5.80	18.84	12.92	16.93	10.05	0.11125785D + 00
2	3.00	15.00	12.00	12.00	7.00	0.56383776D + 00
3	3.00	15.00	12.00	12.00	8.00	0.73077150D − 01
4	3.00	15.00	12.00	12.00	9.00	0.79822798D + 00
5	3.00	15.00	12.00	12.00	10.00	0.11125785D + 00
6	4.00	15.00	15.00	15.00	5.00	**0.72921761D − 10**
7	4.00	15.00	15.00	15.00	6.00	**0.72921760D − 10**
8	4.00	15.00	15.00	15.00	7.00	0.29455572D + 00
9	4.00	15.00	15.00	15.00	8.00	0.29455572D + 00
10	4.00	15.00	15.00	15.00	9.00	0.69999987D + 00
11	5.00	14.00	16.00	16.00	5.00	0.24823123D + 00
12	5.00	14.00	16.00	16.00	6.00	0.24823123D + 00
13	5.00	14.00	16.00	16.00	7.00	**0.28355876D − 04**
14	5.00	14.00	16.00	16.00	8.00	**0.72921764D − 11**
15	5.00	14.00	16.00	16.00	9.00	0.42955934D + 00

$$-1.00 < x(1) < 1.00 \qquad -1.00 < y(1) < 1.00 \qquad 14.00 < x(2) < 16.00$$

$$-2.10 < y(2) < -0.10$$

Case 5: Two additional constraints to the ones imposed in Case 4 have been considered here. The moduli of the velocity and acceleration of point 5 at the time corresponding to the synthesis point 5. If these values for the exact mechanism are $v(5)_5 = 0.50941870\text{D} + 03$; $a(5)_5 = 0.45979694\text{D} + 05$, we shall restrain them to be lower than $v(5)_5 = 0.40000000\text{D} + 03$; $a(5)_5 = 0.42000000\text{D} + 05$, respectively.

The different results obtained are included in Table 4.

Table 4
Final lengths and objective function for each synthesis problem

	Case 1	Case 2	Case 3	Case 4	Case 5
L_1	3.028555	3.72808	3.64680	3.31491	3.73988
L_2	14.59707	14.01445	13.84843	13.56089	16.50489
L_3	15.05072	14.94996	14.57200	14.3975	14.29121
L_4	12.27295	14.16153	15.74827	15.91603	18.63774
L_5	3.55401	5.16000	6.67614	5.70355	8.44462
x_1				1.000000	−0.74363
y_1				0.674266	−1.00000
x_2				16.00000	16.00000
y_2				−0.10000	−0.10000
Objective function	0.1173D + 01	0.6754D − 01	0.1283D + 00	0.4066D − 01	0.1703D + 00

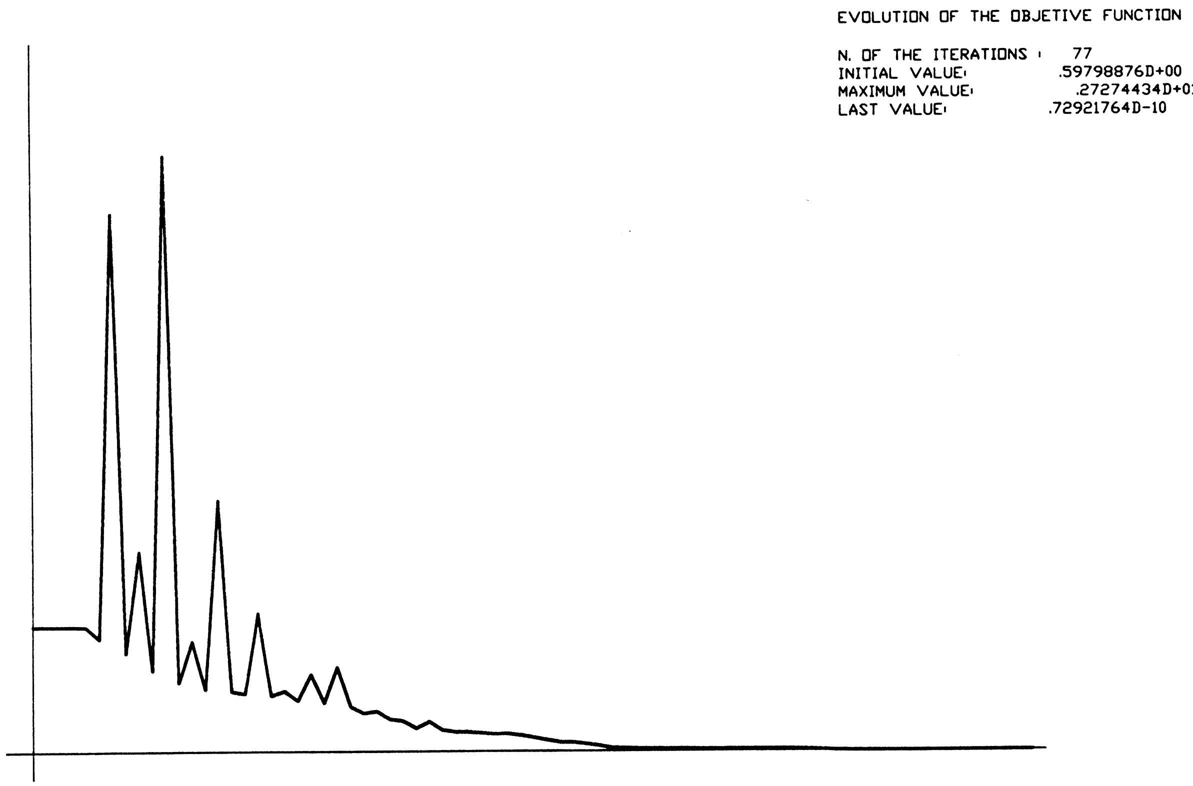

Fig. 7. Evolution of the objective function for the case 13.

As it is easy to observe, this new imposition of constraints implies a certain variation of the bar lengths and an increment of the objective function as it is the difference between the values of the constraints and the exact values of the constrained magnitudes. A certain "mobility" of the fixed points allows the reduction of the objective function, and therefore, the increase of the accuracy of the mechanism with respect to the constraints. Of course, the equality kinematic constraints are always fulfilled.

Table 5
Definition of the synthesis points

SP	*x*(5)	*y*(5)
1	8.500000	−4.830377
2	8.500000	−3.381574
3	8.500000	−1.329634
4	8.500000	0.663212
5	8.500000	2.002536
6	8.500000	4.095180

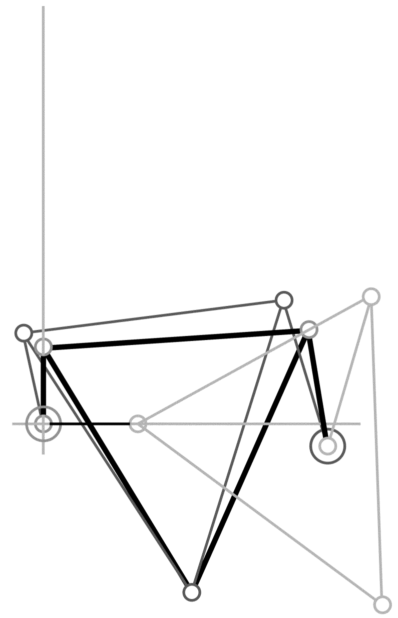

Fig. 8. Different locations of the mechanism 13 along the synthesis process.

4.2. Example 2: the Peaucillier mechanism

This second example studies the position synthesis of a well-known mechanism, the so-called Peaucillier mechanism able to follow a straight vertical path as it is shown in Fig. 9, that as it is established in classical references [10], has to fulfil the following condition

$$L_2 = L_3 \quad L_4 = L_5 = L_6 = L_7 \tag{24}$$

In this case, six points located along the vertical path as shown in Table 5 define the synthesis.

We start from the bar lengths shown in the second column of Table 6 obtaining the final lengths included in the third column of the same table.

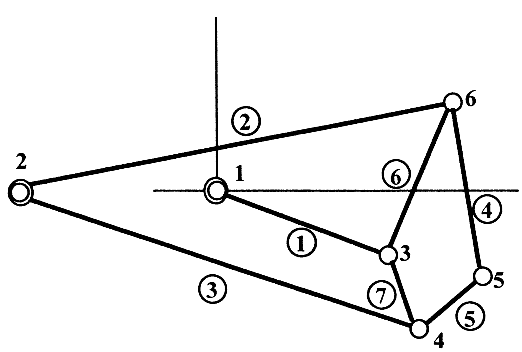

Fig. 9. Peuacillier mechanism.

Table 6
Initial and final bar lengths

Bar	Initial length	Final length
1	5.000000	5.000000
2	10.000000	12.865798
3	14.000000	12.044742
4	5.000000	5.525284
5	2.500000	3.174246
6	6.500000	5.525284
7	3.500000	3.174244

Fig. 10 shows the path of the initial and final mechanism and Fig. 11 the evolution of the objective function.

It is interesting to point out that conditions (24) are only a particular case of the actual ones that may be established as

$$L_4 = L_6 \quad L_5 = L_7 \quad L_2^2 - L_3^2 = L_4^2 - L_5^2 \quad L_2 > L_3 \quad \text{and} \quad L_4 > L_5 \tag{25}$$

that have been here obtained.

4.3. Example 3: box-transporting mechanism

The following example corresponds to a mechanism designed to transport boxes from a certain position to another. The chosen topology is shown in Fig. 12 and the synthesis points

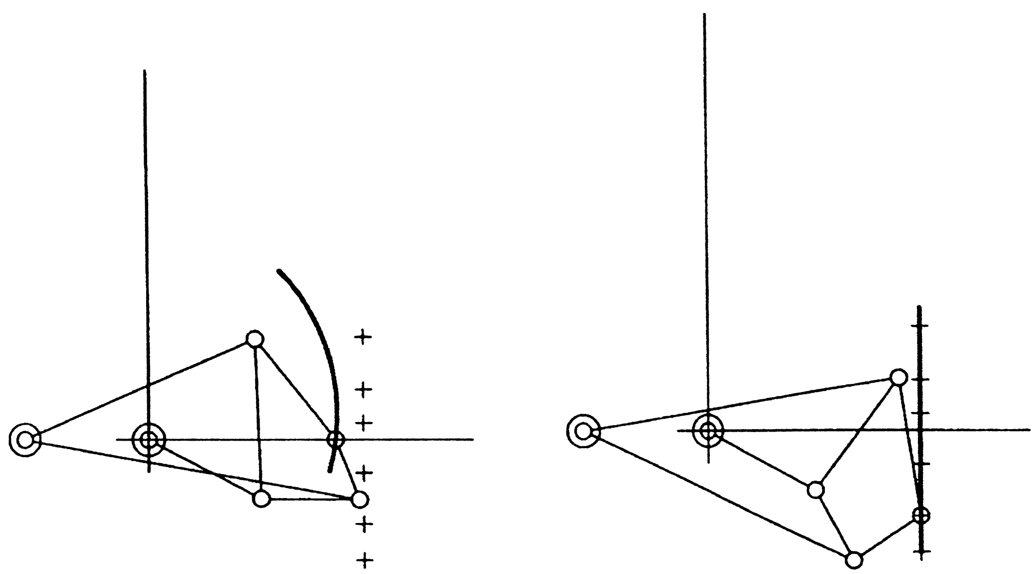

Fig. 10. Intial and final path of the mechanism.

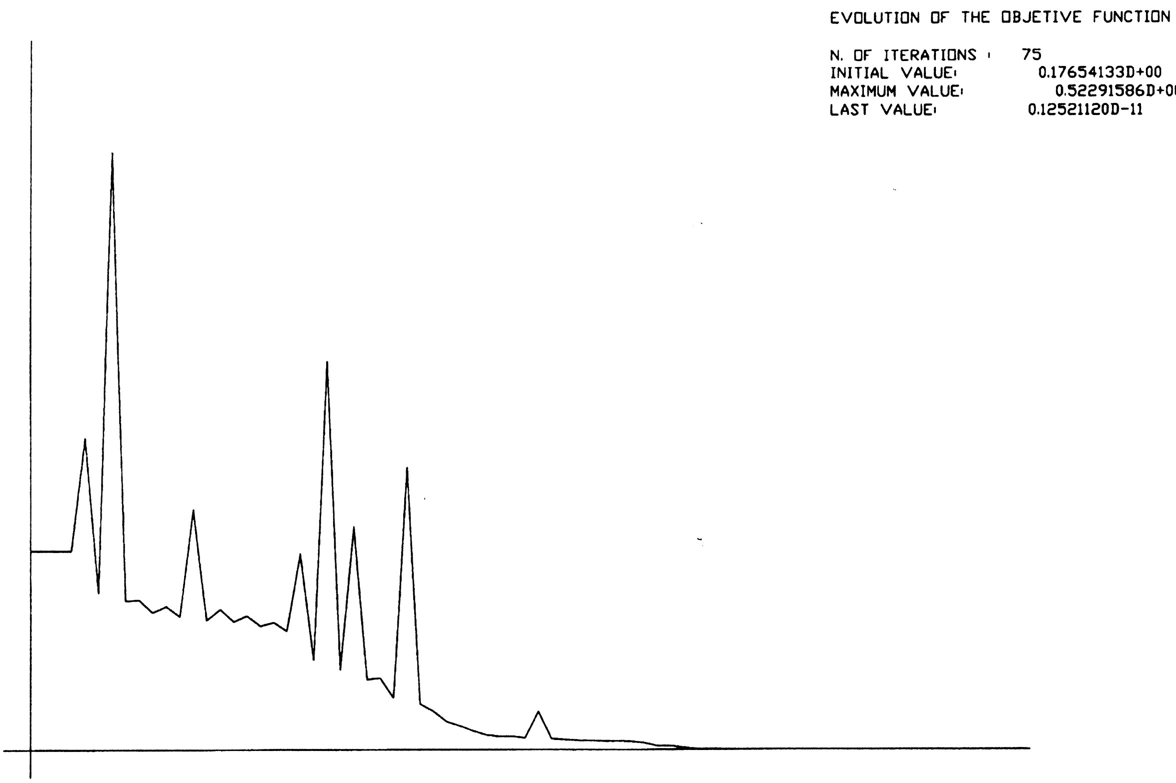

Fig. 11. Evolution of the objective function.

in Table 7. Finally, the initial and final lengths are shown in Table 8. In this case, and on the contrary to the previous examples, the design is not defined by the path of a single point but by the locations of three points in the initial and final positions of the mechanism. Due to the tolerance allowed for these locations, they are constrained to be inside two regions, defined by

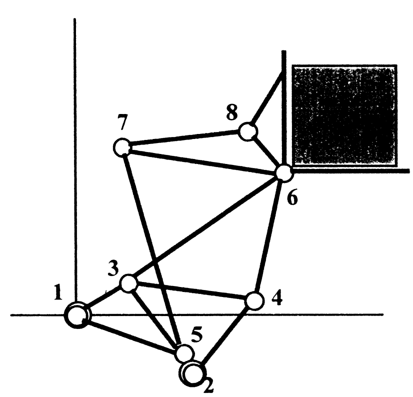

Fig. 12. Proposed box-transporting mechanism.

Table 7
Definition of the synthesis positions (synthesis regions)

Co-ordinates	Constraints	Position 1	Position 2
$x(3)$	Lower band	8.31491600	−1.00000000
	Upper band	8.31491600	1.00000000
$y(3)$	Lower band	3.44415100	−10.00000000
	Upper band	3.44415100	−8.00000000
$x(6)$	Lower band	35.00000000	0.00000000
	Upper band	35.00000000	0.00000000
$y(6)$	Lower band	18.00000000	18.00000000
	Upper band	18.00000000	24.00000000
$x(8)$	Lower band	28.00000000	−9.00000000
	Upper band	28.00000000	−5.00000000
$y(8)$	Lower band	25.14142800	23.14142800
	Upper band	25.14142800	34.14142800

the appropriate constraints bands. The final objective function is 0.13991086D−16, which is not strange since we have chosen the synthesis regions in such a way that they include the exact solution as can be seen in Fig. 13 which also includes the paths of two points of this solution mechanism.

If we now add to the previous constraints, a kinematic one like the null value of the y component of the velocity of node 6 at its initial and final locations, we now get the new solution included in the fourth column of Table 8. This result fulfils both the position and kinematic constraints. Fig. 14 shows the path followed by node 6, being interesting to notice the two cusps appearing in the path as a consequence of the velocity constraints.

Table 8
Initial and final lengths

Bar	Initial length	Final length	Final length
1	9.000000	9.00000	9.00000
2	16.000000	15.909841	15.659176
3	14.000000	14.321545	13.864767
4	36.000000	30.396816	30.396816
5	19.000000	19.936299	17.019403
6	22.000000	19.468617	17.991286
7	18.000000	15.537028	16.822431
8	33.000000	32.719895	32.147923
9	28.000000	28.360886	28.421768
10	22.000000	21.329476	21.421182
11	10.000000	10.000000	10.000000

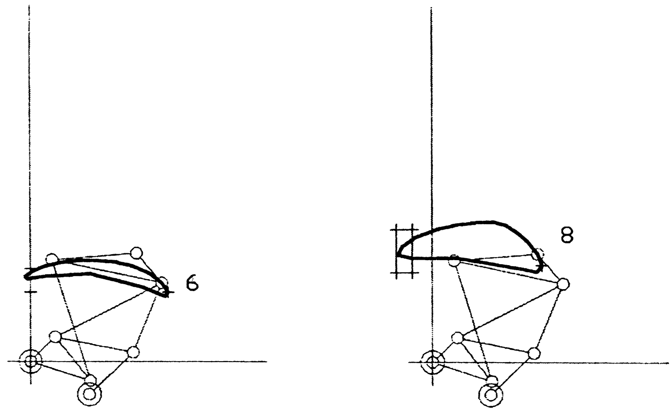

Fig. 13. Path for points 6 and 8.

4.4. Example 4: synthesis with a gear–gear pair

As a final example of 2D mechanisms we study the mechanism shown in Fig. 15, which includes a gear–gear pair. The idea is to solve a functional synthesis problem, establishing a correlation between nodes 3 and 9 of the mechanism. Table 9 defines this functional constraint.

The initial and solution bar lengths are included in the second and third column of Table 10, respectively, while Fig. 16 shows the evolution of the objective function with a final value of 0.29658947D−01.

If we now relax the constraints on node 9 allowing a certain tolerance on its co-ordinates (synthesis region defined by Table 11), we obtain a new mechanism with the lengths included in the fourth column of Table 10. The value of the objective function is now 0.31601815D−10 implying the almost exact fulfilment of the constraints.

We now consider other kinematic constraints like:

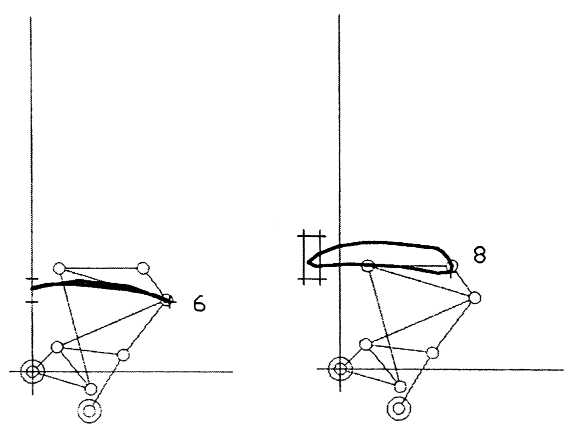

Fig. 14. Final path of points 6 and 8.

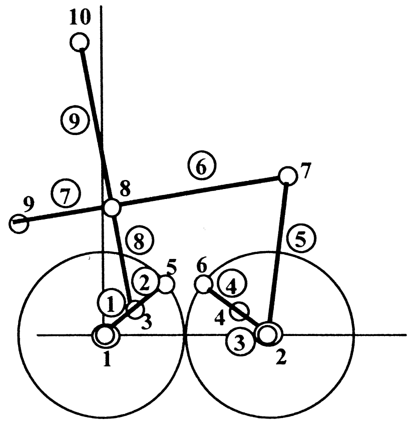

Fig. 15. Gear–gear mechanism.

Case 1: Constraint on velocity $\dot{x}(9)_4 = 1$
Case 2: Constraint on acceleration $\ddot{y}(9)_3 = 0.25\mathrm{E}+04$
Case 3: Constraint on acceleration $\ddot{x}(9)_3 = 0.75\mathrm{E}+04$
Case 4: Constraints on velocity and acceleration $v(9)_4 = 0.35\mathrm{E}+03$; $a(9)_3 = 0.6\mathrm{E}+04$
Case 5: The same as Case 4, but the weighting coefficient for the synthesis point 3 is modified to five times the one of the node 9
Case 6: Constraints on velocity and acceleration including now a node not appearing in the position synthesis definition

$$\dot{x}(9)_3 = -0.2\mathrm{E}+03; \quad \dot{y}(10)_4 = -0.38\mathrm{E}+03; \quad \ddot{x}(9)_3 = 0.75\mathrm{E}+04; \quad \ddot{y}(10)_4 = 0.18\mathrm{E}+04$$

Case 7: Velocity constraints but relaxed by synthesis regions

$$\dot{x}(9)_3 = -0.20\mathrm{E}+03; \quad \dot{y}(10)_6 = [-0.65\mathrm{E}+02, 0.55\mathrm{E}+02];$$

$$\ddot{x}(9)_3 = 0.75\mathrm{E}+04; \quad \ddot{y}(10)_4 = [0.3\mathrm{E}+05, 0.32\mathrm{E}+05]$$

Table 9
Definition of the functional constraints (functional synthesis)

SP	*x*(3)	*y*(3)	*x*(9)	*y*(9)
1	3.695520	1.530730	−28.50000	3.000000
2	1.530730	3.695520	−29.50000	4.900000
3	−1.530730	3.695520	−31.70000	8.100000
4	−4.000000	0.001000	−33.50000	11.200000
5	−2.828430	−2.828430	−33.00000	6.400000
6	0.001000	−4.000000	−31.10000	1.600000
7	3.695520	−1.530730	−29.00000	1.200000

Table 10
Initial and final bar lengths

Bar	Initial length	Final length	Final length
1	4.000000	4.000000	4.000000
2	6.000000	6.017078	5.985318
3	2.000000	5.032170	4.980715
4	8.000000	4.950751	5.033967
5	22.000000	14.580902	14.686802
6	24.000000	17.919763	17.738732
7	14.000000	30.035494	30.051118
8	20.000000	10.390396	10.462367
9	15.000000	9.983679	10.189555

The corresponding results are now shown in Table 12 for each of the proposed cases.

Again we can observe that the addition of kinematic constraints makes it more difficult to obtain a good performance of the mechanism with respect to all of them and eventually the loosing of the accuracy obtained in the functional synthesis. For example, in Case 6, an

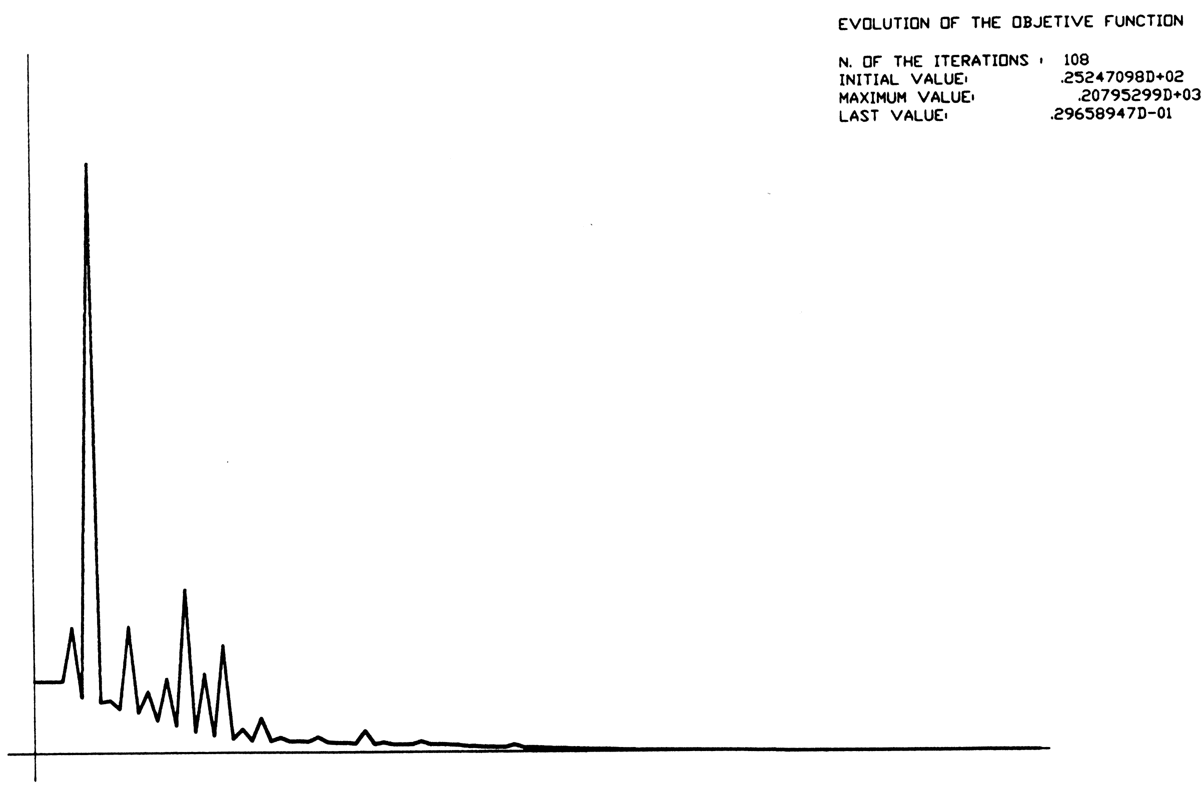

Fig. 16. Evolution of the objective function.

Table 11
Synthesis region for node 9

SP	$x(9)$ Lower	$x(9)$ Upper	$y(9)$ Lower	$y(9)$ Upper
1	−28.60000	−28.40000	2.900000	3.100000
2	−29.60000	−29.40000	4.800000	5.000000
3	−31.80000	−31.60000	8.000000	8.200000
4	−33.60000	−33.40000	11.100000	11.300000
5	−33.10000	−32.90000	6.300000	6.500000
6	−31.20000	−31.00000	1.500000	1.700000
7	−29.10000	−28.90000	1.100000	1.300000

excessively strong constraint gives rise to a very high value of the objective function showing that this new mechanism is not able to fulfil appropriately the proposed functional path.

4.5. Example 5: a three-dimensional problem

Finally, and as an example of three-dimensional mechanism we propose a simple one: a 3D "crank–lever" mechanism with fixed co-ordinates $y(2) = 0.00$; $x(3) = 0.00$; $z(3) = -4.00$ as it is shown in Fig. 17.

We now define the following synthesis points and initial lengths getting the trivial solution given by Tables 13 and 14 with final value for the objective function 0.71651650E−13 that is an exact mechanism, with the evolution of the objective function shown in Fig. 18.

If we now impose additional kinematic constraints and allow a more relaxed location region for both nodes 2 and 3, defined by Table 15, we get, for the following cases, the results given in Table 16:

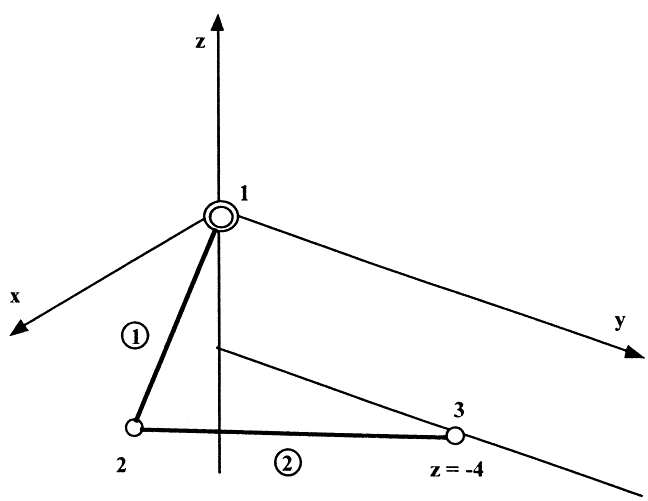

Fig. 17. 3D crank–lever mechanism.

Table 12
Final length for each bar and each case

Bar	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
L1	4.00000	4.00000	4.00000	4.00000	4.00000	4.00000	4.00000
L2	6.00425	5.99451	5.97130	6.12413	6.04618	5.98532	5.55492
L3	5.01846	5.05275	5.06605	5.16573	5.23969	4.98071	4.15610
L4	4.97729	4.95273	4.96265	4.71014	4.71334	5.03397	6.28898
L5	15.34750	14.56018	15.3539	15.22337	15.17334	14.68680	17.58223
L6	17.65937	17.85362	17.86493	18.64378	18.91315	17.73873	15.36328
L7	30.10148	30.04307	30.01498	30.02815	30.03041	30.05103	29.99511
L8	10.91440	10.36011	10.96530	10.89495	10.93462	10.46237	12.37788
L9	10.31390	10.21663	9.15978	9.54994	9.53201	10.18955	6.75044
Objective function	0.1085D−02	0.1677D−10	0.3599D−01	0.1783D+00	0.3571D+00	0.1356D+05	0.6538D+00

Table 13
Definition of the synthesis

SP	$x(2)$	$z(2)$	$y(3)$
1	2.000000	0.000000	6.633250
2	0.765367	1.847759	5.450536

Table 14
Bar lengths

Bar	Initial length	Final length
1	1.0000	2.0000
2	9.0000	8.0000

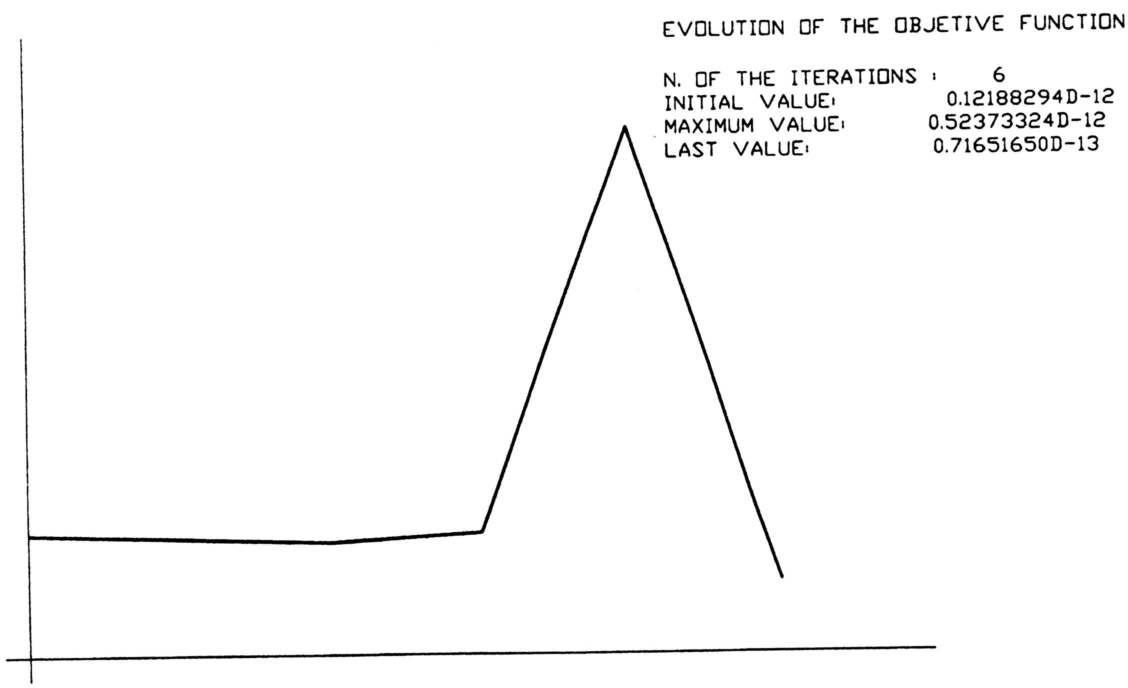

Fig. 18. Evolution of the objective function.

Table 15
Synthesis regions for kinematic constraint

Co-ordinate		Node 1	Node 2
$y(3)$	Lower bound	4.633250	3.405361
	Upper bound	8.633250	7.405361

Case 1: Constraint on velocity: $\dot{y}(3)_1 = 100$
Case 2: Constraint on acceleration: $\ddot{y}(3)_2 = 10{,}000$
Case 3: Constraint on velocity and acceleration: $\dot{y}(3)_1 = 100$; $\ddot{y}(3)_2 = 10{,}000$

It is clear that the first two cases are able to obtain the exact mechanism, while the third only approximates the constraints which is understandable since the two constraints included in Case 3 are the same than in Cases 1 and 2 which gave rise to different mechanisms being, therefore, impossible to obtain a unique mechanism able of fulfilling both at the same time.

5. Conclusions

In this paper, we have presented a method for the optimal synthesis of mechanisms that, on the contrary to other previously proposed methods, considers as design variables all the different co-ordinates (natural co-ordinates) which define the location and kinematics of the mechanism. This is established for all the different times at whom a certain synthesis condition is imposed, together with the lengths of each of the topological components of the mechanism. This allows the complete control of the mechanism for each of those times, without the necessity of approximating the gradient of the objective function and/or constraints that now can be obtained analytically and very easily for most of the usual constraints. The price for this simplicity is the computational cost of the problem due to the large number of design variables when having a big number of synthesis conditions. Anyhow, due to the exponential increment of computer performance and the usually small number of elements that compose most of the practical mechanisms, this seems to be a very small price when compared to the above advantages including its generality (the extension to 3D situations is straightforward without including any new concept or special formulation) and reliability.

This simple analytical treatment both of constraints and objective functions allows to use a very robust optimisation method like the sequence of quadratic problems. This is accomplished

Table 16
Different results for kinematic constraints

Case	L_1	L_2	$y(3)_1$	$y(3)_2$	Objective function
1	2.0	9.165151	8.0000000	7.015549	0.34507278D−16
2	2.0	9.315438	8.1717430	7.210778	0.34507279D−16
3	2.0	9.240195	8.0000000	7.210778	0.11293050D−01

without any complexity in programming, being very easy for the user to include any additional constraint with the only effort of formulating the gradient of that constraint with respect to the design variables.

The possibility of considering different weighting coefficients for each constraint and the possibility of defining regions of possible locations of each of the nodes of the mechanism allow to consider very easily useful design directives. These may be established, for instance, according to the importance of the change of length of each bar (production or building constraints may be formulated usually in this way). This implies a strong flexibility in the design process which, together with reliability and generality, is one of the most important properties of this kind of packages.

Also, the fact that the design vector includes all the degrees of freedom of the problem for all the times of interest allows the addition of kinematic constraint with a very small extension of the program as has been shown in the examples and theoretical development above. This allows the establishment of very usual functional constraints, like limits on the value of the velocity at certain critical points of the path, which is very important, for instance, to limit impacts, or the possibility of establishing limits on the value of the acceleration usually related to the need of having small inertia forces when manipulating materials with low resistance.

This tool is also very efficient in detecting easily, interferences, impossible paths, etc. on the proposed mechanism or topology with no more than observing the value of the objective function at the end of the synthesis. Anyhow a deep study of the mechanism prior to the start of the synthesis process is very useful to detect also this kind of problems and establish appropriate values for the bands bounding the constraints or for the initial iteration vectors, although it has been proved that this approach is robust enough to deal with initial iterations vectors far enough from the actual solution, keeping of course the same mechanism topology.

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