SHORT TERM EVOLUTION OF ARTIFICIAL SATELLITES

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Abstract. When the elimination of the parallax and the elimination of the perigee is applied to the zonal problem of the artificial satellite, a one-degree of freedom Hamiltonian is obtained. The classical way to integrate this Hamiltonian is by applying the Delaunay normalization, however, changing the time to the perturbed true anomaly and the variable to the inverse of the distance, the Hamilton equations become a perturbed harmonic oscillator. In this paper we apply the Krylov–Bogoliubov–Mitropolsky (KBM) method to integrate the perturbed harmonic oscillator as an alternative method to the Delaunay normalization. This method has no problem with small eccentricities and inclinations, and shows good numerical results in the evaluation of ephemeris of satellites.

Key words: artificial satellite, harmonic oscillators, KBM method

1. Introduction

Delaunay normalization is one of the most used methods when dealing with analytical theories of the artificial satellite. The first attempt to the main problem of the artificial satellite was due to Brouwer (1959). Some years later, Coffey and Deprit (1982) presented a completely analytical closed form third order solution to the same problem. The method proposed by Coffey and Deprit for Delaunay normalization avoids series expansion in the eccentricity by taking advantage of the invariance of Poisson Brackets with respect to the canonical variables, and using Whittaker's variables instead of the explicit use of Delaunay's variables.

The solution of Coffey and Deprit has two advantages over the Brouwer's one. On the one hand, their theory is valid until the third order for all range of values of the eccentricity. Singularities of small eccentricities and inclinations can be avoided when expressing the problem by a set of regular variables such as those defined by Deprit and Rom (1970). On the other hand, instead of using the Poincaré method, they applied two Lie transformations: the elimination of the parallax, (Deprit, 1981), and the Delaunay normalization. With this scheme they reduce the number of terms of the generators, achieving a more compact theory and the drastic reduction in computational times required in orbit prediction. A new simplification, the elimination of the perigee (Alfriend and Coffey, 1984), was introduced later on to improve the previous theory.

To obtain the generating function in the method of Coffey and Deprit, it is necessary a way to decompose a function as a direct sum of a kernel's function

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and the image of the Lie derivative. This presents difficulties with certain kind of functions when they are not explicitly expressed in Delaunay functions; as a matter of fact, Coffey and Deprit pointed out in their work the appearance of groups of terms that require individual consideration in order to obtain the generator. Part of the integration was made using a partial data base of integrals instead of using a general procedure, thus the extension of the method to a higher order is not guaranteed. A detailed study of the decomposition of functions appears in the paper of Osacar and Palacián (1994). They prove the appearance of special functions like dilogarithmic functions in this process.

Coffey and Alfriend (1984) wrote AOPP, an analytical orbit prediction program generator written in FORTRAN, to obtain automatically, by computer, analytical theories of the artificial satellite including the elimination of the parallax, the elimination of the perigee and Delaunay normalization. They used the Poisson Series Processor of Dassenbrock (1983), also a FORTRAN program, and took only into account the zonal harmonics in the Geopotential. Later on, Palacián (1992) wrote MALISIAS, a *Mathematica* package that extends the idea of AOPP.

Using the C language, more accomplished than FORTRAN or *Mathematica* to handle symbolically Poisson Series, we wrote firstly PSPC (Abad and San Juan, 1993), a new Poisson Series Processor, and later ATESAT (San Juan, 1994; Abad and San Juan, 1995; San Juan, 1996; Abad et al., 1998), a software tool for obtaining automatically ephemeris from analytical simplifications. ATESAT is similar in conception to AOPP and MALISIAS, but the use of C and the improvements in handling Poisson Series introduced by PSPC make ATESAT a more efficient tool than its predecessors. One substantial improvement in ATESAT is the automatic generation of a C-code to evaluate the theories. This code gives us an easy way to obtain ephemeris of a particular satellite from an analytical theory, and to test the accuracy of such theory applied to a particular satellite.

The new possibilities opened by ATESAT in generating high orders of the theories, suggested us the possibility of introducing an alternative to the Delaunay normalization. In this way, inspecting the expression of the Hamiltonian function after the elimination of the perigee, we can see a Hamiltonian with one degree of freedom which is easily converted into a perturbed harmonic oscillator by means of a change of variable and time. We choose the Krylov–Bogoliubov–Mitropolsky (KBM) technique to integrate this oscillator.

Krylov and Bogoliubov (1947) developed a method to obtain the solution of the equation $\ddot{u} + \omega^2 u = \epsilon F(u, \dot{u}; \epsilon)$ to any order. This technique was amplified and justified by Bogoliubov and Mitropolsky (1961). The method was firstly applied by Calvo (1971) to the main problem of the satellite and by Caballero (1975) to a model including the J_2, J_3, J_4 harmonics; in both cases after an application of the Von–Zeipel method to eliminate θ. Later on Sein–Echaluce (1986) applied the KBM method to the integration of radial intermediaries in the satellite theory. We present here a different formulation than the original method better adapted to the symbolic computation.

The method proposed here, that has been included as an option in ATESAT, can be applied to the zonal problem. Eventually, applying the elimination of the parallax and the perigee and taking as new variable the inverse of the distance $1/r$ and the true anomaly as a new time, we end up with an equation ready to be applied to the KBM method. We illustrate the method by obtaining a second order closed theory of the main problem of the satellite; however, ATESAT is capable to obtain theories of any order by choosing any model of zonal problem. In fact, the numerical results showed in the last section of this paper have been obtained by using the ephemeris programs generated by ATESAT after integrating the main problem until third and fourth order.

2. Elimination of the Parallax and Perigee

The Hamiltonian of the *main problem* when expressed in Whittaker's or polar-nodal variables $(r, \theta, \nu, R, \Theta, N)$ is (see Deprit, 1981, for details)

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1,$$

where

$$\mathcal{H}_0 = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - \omega N,$$

$$\mathcal{H}_1 = \frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 P_2(\sin i \sin\theta),$$

and the small parameter $\epsilon = J_2$.

Here, we are only considering the main problem of the satellite theory, but the results can be extended to any zonal problem.

The first step of this theory is to apply the elimination of the parallax. This elimination reduces the complexity of the later calculations when going up to higher orders. Besides, the elimination of the parallax algorithm permits to compute the expression in closed form of the eccentricity and therefore, to obtain theories of general purpose, valid for any kind of elliptic eccentricity (see Deprit (1981), for further details).

After the parallax elimination, the Hamiltonian has the expression

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} + \epsilon\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^2\left(\frac{1}{2} - \frac{3}{4}\sin^2 i\right) + \\ & + \frac{\epsilon^2}{2!}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^4\left[-\frac{5}{4} - \frac{3}{8}C^2 - \frac{3}{8}S^2 + \right. \\ & \left. + \left(\frac{21}{8} + \frac{27}{16}C^2 - \frac{15}{16}S^2\right)\sin^2 i + \left(-\frac{21}{16} - \frac{75}{64}C^2 + \frac{105}{64}S^2\right)\sin^4 i\right],\end{aligned} \tag{1}$$

where $C = e\cos g$ and $S = e\sin g$.

The elimination of the perigee (Alfriend and Coffey, 1984) is applied to remove the argument of the perigee g from the perturbation. This elimination is not a normalization in the sense that the transformed Hamiltonian does not belong to the kernel of the Lie derivative $\mathcal{L}_0$ associated with $\mathcal{H}_0$, but it reduces by one the number of degrees of freedom.

Finally, after the two previous transformations, the second order Hamiltonian becomes

$$\begin{aligned}\mathcal{H} \;=\; & \frac{1}{2}\left(R^2+\frac{\Theta^2}{r^2}\right)-\frac{\mu}{r}+\epsilon\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^2\left(\frac{1}{2}-\frac{3}{4}\sin^2 i\right)+\\ & +\frac{\epsilon^2}{2!}\left[\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^4\left(-\frac{13}{8}+3\sin^2 i-\frac{69}{64}\sin^4 i\right)+\right.\\ & \left.+\;\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^4\eta^2\left(\frac{3}{8}-\frac{3}{8}\sin^2 i-\frac{15}{64}\sin^4 i\right)\right]. \qquad (2)\end{aligned}$$

Substituting η, p by their values in function of polar–nodal variables

$$\eta^2=\frac{2p}{r}-\frac{p^2}{r^2}-\frac{R^2p^2}{\Theta^2},\qquad p=\frac{\Theta^2}{\mu},$$

we find

$$\begin{aligned}\mathcal{H} \;=\; & \frac{1}{2}\left(R^2+\frac{\Theta^2}{r^2}\right)-\frac{\mu}{r}+\epsilon\mathcal{M}_1^{0,2}\frac{1}{r^2}+\\ & +\frac{\epsilon^2}{2!}\left(\mathcal{M}_2^{0,2}\frac{1}{r^2}+\mathcal{M}_2^{0,3}\frac{1}{r^3}+\mathcal{M}_2^{0,4}\frac{1}{r^4}+\mathcal{M}_2^{2,2}\frac{R^2}{r^2}\right), \qquad (3)\end{aligned}$$

where $\mathcal{M}_n^{i,j}$ represent the coefficients of R^i/r^j at order n, and they are functions of the constants α, μ and the momenta Θ, N. In particular

$$\begin{aligned}\mathcal{M}_1^{0,2} &= \frac{\alpha^2\mu^2}{\Theta^2}\left(\frac{1}{2}-\frac{3}{4}\sin^2 i\right),\\ \mathcal{M}_2^{0,2} &= \frac{\alpha^4\mu^4}{\Theta^6}\left(-\frac{13}{8}+3\sin^2 i-\frac{69}{64}\sin^4 i\right),\\ \mathcal{M}_2^{0,3} &= \frac{\alpha^4\mu^3}{\Theta^4}\left(\frac{3}{4}-\frac{3}{4}\sin^2 i-\frac{15}{32}\sin^4 i\right),\\ \mathcal{M}_2^{0,4} &= \frac{\alpha^4\mu^2}{\Theta^2}\left(-\frac{3}{8}+\frac{3}{8}\sin^2 i+\frac{15}{64}\sin^4 i\right),\\ \mathcal{M}_2^{2,2} &= \frac{\alpha^4\mu^2}{\Theta^4}\left(-\frac{3}{8}+\frac{3}{8}\sin^2 i+\frac{15}{64}\sin^4 i\right).\end{aligned}$$

3. Radial Distance and Radial Velocity as a Perturbed Harmonic Oscillator

Taking into account the expression of the Hamiltonian (3), the Hamilton equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial R}, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial r},$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\Theta}, \qquad \frac{\mathrm{d}\Theta}{\mathrm{d}t} = 0,$$
$$\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial N}, \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = 0,$$

show us that the momenta Θ, N are constants, and the problem is transformed into computing two quadratures to obtain θ, ν and solving the first two differential equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = R + \frac{\epsilon^2}{2!}\left(2\mathcal{M}_2^{2,2}\frac{R}{r^2}\right), \tag{4}$$

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= \frac{\Theta^2}{r^3} - \frac{\mu}{r^2} + \epsilon\left(2\mathcal{M}_1^{0,2}\frac{1}{r^3}\right) + \\ &\quad + \frac{\epsilon^2}{2!}\left(2\mathcal{M}_2^{0,2}\frac{1}{r^3} + 3\mathcal{M}_2^{0,3}\frac{1}{r^4} + 4\mathcal{M}_2^{0,4}\frac{1}{r^5} + 2\mathcal{M}_2^{2,2}\frac{R^2}{r^3}\right).\end{aligned} \tag{5}$$

Using the algorithm of series inversion given in the appendix, with the Equation (4), we obtain the expression of R in function of r and $\mathrm{d}r/\mathrm{d}t$ in the form

$$R = \frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\epsilon^2}{2!}\left(-2\mathcal{M}_2^{2,2}\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\right). \tag{6}$$

This algorithm is particularly useful for higher orders, where powers of R greater than one appear in the expression (4).

Differentiating again (4), we have

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \frac{\mathrm{d}R}{\mathrm{d}t} + \frac{\epsilon^2}{2!}2\mathcal{M}_2^{2,2}\left(\frac{1}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t} - 2\frac{R}{r^3}\frac{\mathrm{d}r}{\mathrm{d}t}\right), \tag{7}$$

and eventually, substituting (5) and (6) into (7) we obtain the second order differential equation

$$\begin{aligned}\frac{\mathrm{d}^2r}{\mathrm{d}t^2} &= \frac{\Theta^2}{r^3} - \frac{\mu}{r^2} + \epsilon\left(2\mathcal{M}_1^{0,2}\frac{1}{r^3}\right) + \frac{\epsilon^2}{2!}\left[2\mathcal{M}_2^{0,2}\frac{1}{r^3} + 3\mathcal{M}_2^{0,3}\frac{1}{r^4} + 4\mathcal{M}_2^{0,4}\frac{1}{r^5}\right. \\ &\quad \left. + \, 2\mathcal{M}_2^{2,2}\frac{\Theta^2}{r^5} - 2\mathcal{M}_2^{2,2}\frac{\mu}{r^4} - 2\mathcal{M}_2^{2,2}\frac{1}{r^3}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right].\end{aligned} \tag{8}$$

Finally, we will change the variables r, $\mathrm{d}r/\mathrm{d}t$ and the time t by two new variables u, v and a new time s defined by

$$u = \frac{1}{r} + \beta, \qquad r^2\frac{\mathrm{d}s}{\mathrm{d}t} = \Theta, \qquad v = \frac{\mathrm{d}u}{\mathrm{d}s}, \tag{9}$$

where we call

$$\beta = \mu/\Theta^2. \tag{10}$$

These equations together with the differential relations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\Theta\, v, \qquad \frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\Theta^2\,(u-\beta)^2\,\frac{\mathrm{d}^2u}{\mathrm{d}s^2}, \tag{11}$$

transform (8) into the equation of a perturbed harmonic oscillator

$$\begin{aligned}\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + u \;=\; & \epsilon(\mathcal{K}_1^{0,0} + u\mathcal{K}_1^{1,0}) + \frac{\epsilon^2}{2!}[\mathcal{K}_2^{0,0} + u\mathcal{K}_2^{1,0} + u^2\mathcal{K}_2^{2,0} + \\ & + u^3\mathcal{K}_2^{3,0} + v^2\mathcal{K}_2^{0,2} + uv^2\mathcal{K}_2^{1,2}],\end{aligned} \tag{12}$$

where $\mathcal{K}_n^{i,j}$ represent the coefficients of $u^i v^j$ at order n, given by

$$\begin{aligned}\mathcal{K}_1^{0,0} &= -\frac{2\mu\mathcal{M}_1^{0,2}}{\Theta^4},\\ \mathcal{K}_1^{1,0} &= -\frac{2\mathcal{M}_1^{0,2}}{\Theta^2},\\ \mathcal{K}_2^{0,0} &= -\frac{2\mu\mathcal{M}_2^{0,2}}{\Theta^4} - \frac{3\mu^2\mathcal{M}_2^{0,3}}{\Theta^6} - \frac{4\mu^3\mathcal{M}_2^{0,4}}{\Theta^8},\\ \mathcal{K}_2^{1,0} &= -\frac{2\mu^2\mathcal{M}_2^{2,2}}{\Theta^4} - \frac{2\mathcal{M}_2^{0,2}}{\Theta^2} - \frac{6\mu\mathcal{M}_2^{0,3}}{\Theta^4} - \frac{12\mu^2\mathcal{M}_2^{0,4}}{\Theta^6},\\ \mathcal{K}_2^{2,0} &= -\frac{4\mu\mathcal{M}_2^{2,2}}{\Theta^2} - \frac{3\mathcal{M}_2^{0,3}}{\Theta^2} - \frac{12\mu\mathcal{M}_2^{0,4}}{\Theta^4},\\ \mathcal{K}_2^{3,0} &= -2\mathcal{M}_2^{2,2} - \frac{4\mathcal{M}_2^{0,4}}{\Theta^2},\\ \mathcal{K}_2^{0,2} &= \frac{2\mu\mathcal{M}_2^{2,2}}{\Theta^2},\\ \mathcal{K}_2^{1,2} &= 2\mathcal{M}_2^{2,2}.\end{aligned} \tag{13}$$

4. Krylov–Bogoliubov-Mitropolsky Method

The Krylov–Bogoliubov-Mitropolski (KBM) method obtains asymptotic approximations for the general weakly nonlinear second order equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \omega^2 u = F(u, v; \epsilon), \tag{14}$$

where

$$v = \frac{\mathrm{d}u}{\mathrm{d}t},$$

ϵ is a small parameter and F can be expanded as a power series of ϵ in the form

$$F(u, v; \epsilon) = \epsilon f(u, v; \epsilon) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} F_{n,0}(u, v), \quad F_{0,0}(u, v) = 0. \tag{15}$$

When $\epsilon = 0$, the solution of (14) can be written as $u = \delta\cos\psi$ with a constant amplitude δ and a uniformly rotating phase angle $\psi = \omega t + \psi_0$. To determine an approximate solution to (14) for ϵ different from zero, Krylov and Bogoliubov (1947), developed a technique, improved and justified later by Bogoliubov and Mitropolski (1961). They assumed an asymptotic expansion of the solution in the form

$$u = \delta\cos\psi + \sum_{n\geqslant 1} \frac{\epsilon^n}{n!} u_n(\delta, \psi) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} u_n(\delta, \psi), \quad u_0 = \delta\cos\psi, \tag{16}$$

where each $u_n(\delta, \psi)$ is a 2π-periodic function of ψ, and δ assumed to vary with time according to

$$\begin{aligned} \frac{\mathrm{d}\delta}{\mathrm{d}t} &= \sum_{n\geqslant 1} \frac{\epsilon^n}{n!} A_n(\delta) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} A_n(\delta), \quad A_0 = 0, \\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= \omega + \sum_{n\geqslant 1} \frac{\epsilon^n}{n!} B_n(\delta) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} B_n(\delta), \quad B_0 = \omega. \end{aligned} \tag{17}$$

From here on, we will make use of a generalization of the Cauchy formula

$$\prod_{n=1}^{k} \left(\sum_{i\geqslant 0} \frac{\epsilon^i}{i!} a_i^{(n)} \right) = \sum_{i\geqslant 0} \frac{\epsilon^i}{i!} \left(\sum_{j_1+j_2+\cdots+j_k} \frac{i!}{j_1!j_2!\ldots j_k!} a_{j_1}^{(1)} a_{j_2}^{(2)} \ldots a_{j_k}^{(k)} \right). \tag{18}$$

Differentiating (16) with respect to time, we have

$$v = \frac{\mathrm{d}u}{\mathrm{d}t} = \left(\sum_{n\geqslant 0} \frac{\epsilon^n}{n!} \frac{\partial u_n}{\partial \delta} \right) \frac{\mathrm{d}\delta}{\mathrm{d}t} + \left(\sum_{n\geqslant 0} \frac{\epsilon^n}{n!} \frac{\partial u_n}{\partial \psi} \right) \frac{\mathrm{d}\psi}{\mathrm{d}t},$$

where substituting (17) and applying (18), we find

$$v = \sum_{n \geqslant 0} \frac{\epsilon^n}{n!} v_n(\delta, \psi), \qquad v_n(\delta, \psi) = \sum_{i+j=n} \frac{n!}{i!j!} \left(A_i \frac{\partial u_j}{\partial \delta} + B_i \frac{\partial u_j}{\partial \psi} \right). \quad (19)$$

In particular, $v_0 = -\omega\delta \sin\psi$.

Differentiating again v with respect to time, we can write

$$\begin{aligned} \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega^2 u \;=\; & \sum_{n \geqslant 0} \frac{\epsilon^n}{n!} \left[\omega^2 u_n + \sum_{i+j+k=n} \frac{n!}{i!j!k!} \left(A_i \frac{\mathrm{d}A_j}{\mathrm{d}\delta} \frac{\partial u_k}{\partial \delta} + \right.\right. \\ & + \left.\left. A_i \frac{\mathrm{d}B_j}{\mathrm{d}\delta} \frac{\partial u_k}{\partial \psi} + A_i A_j \frac{\partial^2 u_k}{\partial \delta^2} + B_i B_j \frac{\partial^2 u_k}{\partial \psi^2} + 2 A_i B_j \frac{\partial^2 u_k}{\partial \delta\, \partial \psi} \right) \right]. \end{aligned} \quad (20)$$

Considering the values of u_0, A_0, B_0, and inspecting in (20) all the terms with i, j or k equal to n, we find the following non-zero terms:

$$i = n, \quad j = 0, \quad k = 0 \rightarrow 2A_i B_j \frac{\partial^2 u_k}{\partial \delta\, \partial \psi} = 2A_n B_0 \frac{\partial^2 u_0}{\partial \delta\, \partial \psi} = -2\omega A_n \sin\psi,$$

$$i = n, \quad j = 0, \quad k = 0 \rightarrow B_i B_j \frac{\partial^2 u_k}{\partial \psi^2} = B_n B_0 \frac{\partial^2 u_0}{\partial \psi^2} = -\omega B_n \delta \cos\psi,$$

$$i = 0, \quad j = n, \quad k = 0 \rightarrow B_i B_j \frac{\partial^2 u_k}{\partial \psi^2} = B_0 B_n \frac{\partial^2 u_0}{\partial \psi^2} = -\omega B_n \delta \cos\psi,$$

$$i = 0, \quad j = 0, \quad k = n \rightarrow B_i B_j \frac{\partial^2 u_k}{\partial \psi^2} = B_0 B_0 \frac{\partial^2 u_n}{\partial \psi^2} = \omega^2 \frac{\partial^2 u_n}{\partial \psi^2}.$$

Thus, we may write

$$\begin{aligned} \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega^2 u \;=\; & \sum_{n \geqslant 0} \frac{\epsilon^n}{n!} \Big(\omega^2 \frac{\partial^2 u_n}{\partial \psi^2} + \omega^2 u_n - \\ & - 2\omega A_n \sin\psi - 2\omega B_n \delta \cos\psi + w_n \Big), \end{aligned} \quad (21)$$

with

$$\begin{aligned} w_n \;=\; & \sum_{\substack{i+j+k=n \\ i,j,k \neq n}} \frac{n!}{i!j!k!} \left(A_i \frac{\mathrm{d}A_j}{\mathrm{d}\delta} \frac{\partial u_k}{\partial \delta} + A_i \frac{\mathrm{d}B_j}{\mathrm{d}\delta} \frac{\partial u_k}{\partial \psi} + \right. \\ & + \left. A_i A_j \frac{\partial^2 u_k}{\partial \delta^2} + B_i B_j \frac{\partial^2 u_k}{\partial \psi^2} + 2A_i B_j \frac{\partial^2 u_k}{\partial \delta\, \partial \psi} \right). \end{aligned} \quad (22)$$

The expression (21) represents the left-hand side of (14) in terms of δ, ψ. The right-hand side can be obtained by expanding in power series of ϵ the function $F(u, v; \epsilon)$ in the form

$$F(u, v; \epsilon) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} F_{n,0} \left(\sum_{n\geqslant 0} \frac{\epsilon^n}{n!} u_n, \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} v_n \right) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} F_{0,n}, \tag{23}$$

where $F_{0,n} = F_{0,n}(u_o, u_1, \ldots, u_{n-1}, v_o, v_1, \ldots, v_{n-1})$.

Equating each order of the right-hand sides of Equations (21) and (23), for each $n \geqslant 1$ yields

$$\begin{aligned} &\omega^2 \left(\frac{\partial^2 u_n}{\partial \psi^2} + u_n \right) \\ &\quad = 2\omega A_n \sin\psi + 2\omega B_n \delta \cos\psi + U_n, \quad U_n = F_{0,n} - w_n, \end{aligned} \tag{24}$$

where U_n depends only on $A_p, B_p, u_p, p \leqslant n-1$.

Let us suppose now a 2π-periodic function $f(\psi)$. It will admit a Fourier series expansion in the form

$$f(\psi) = c_0(f) + \sum_{j\geqslant 1} c_j(f) \cos j\psi + s_j(f) \sin j\psi. \tag{25}$$

Substituting u_n, U_n in (24) by their Fourier expansions and equating term by term we obtain

$$A_n = -\frac{s_1(U_n)}{2\omega}, \qquad B_n = -\frac{c_1(U_n)}{2\omega\delta}, \tag{26}$$

and the coefficients of the expansion of u_n in the form

$$\begin{aligned} &c_0(u_n) = \frac{c_0(U_n)}{\omega^2}, \qquad c_j(u_n) = \frac{c_j(U_n)}{\omega^2(1-j^2)}, \\ &s_j(u_n) = \frac{s_j(U_n)}{\omega^2(1-j^2)}, \quad j \geqslant 2. \end{aligned} \tag{27}$$

Note that the coefficients $c_1(u_n)$, $s_1(u_n)$ remain undetermined and can be chosen to be equal to zero. The last expressions give u_n, A_n, B_n from U_n and permit to obtain the Fourier series expansion of U_{n+1} and continue to the next order.

To complete the method we need to obtain the initial constants $\delta(t_0)$, $\psi(t_0)$ from the values $u(t_0)$, $v(t_0)$. In order to do that we invert the Equations (16) and (19) by using the algorithm given in the appendix.

To apply the algorithm in our problem, we will take into account that $u_0 = \delta \cos\psi$, $v_0 = -\omega\delta \sin\psi$ and the functions u_n, v_n are Fourier expansions of the angular variable ψ. Then, defining C, S by means of the expressions

$$C = \delta \cos\psi, \qquad S = -\omega\delta \sin\psi, \tag{28}$$

we may write

$$\delta = \frac{\sqrt{\omega^2 C^2 + S^2}}{\omega}, \qquad \cos\psi = \frac{C\,\omega}{\sqrt{\omega^2 C^2 + S^2}},$$
$$\sin\psi = -\frac{S}{\sqrt{\omega^2 C^2 + S^2}}, \tag{29}$$

and using the definitions of the Chebyshev polynomials

$$\cos n\psi = T_n(\cos\psi), \qquad \sin n\psi = \sin\psi\, U_{n-1}(\cos\psi), \tag{30}$$

we have

$$\begin{aligned} \cos n\psi &= T_n\left(\frac{C\,\omega}{\sqrt{\omega^2 C^2 + S^2}}\right), \\ \sin n\psi &= \frac{-S}{\sqrt{\omega^2 C^2 + S^2}}\, U_{n-1}\left(\frac{C\,\omega}{\sqrt{\omega^2 C^2 + S^2}}\right). \end{aligned} \tag{31}$$

Applying these relation to the Equations (16) and (17) we finally obtain

$$u = C + \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} u_n(C,S), \qquad v = S + \sum_{n\geqslant 0} \frac{\epsilon^n}{n!} v_n(C,S). \tag{32}$$

Eventually, the inversion algorithm, given in the appendix, permits to obtain explicitly C, S in terms of u, v, and consequently δ, ψ.

5. Time Variation of *r* and *R*

Applying the method seen in the previous Section 4 to the differential Equation (12) we obtain, until second order, the following solution

$$\begin{aligned} u &= \delta\cos f + \epsilon\mathcal{K}_1^{0,0} + \\ &+ \frac{\epsilon^2}{2!}\Bigg[\Big\{2\mathcal{K}_1^{0,0}\mathcal{K}_1^{1,0} + \mathcal{K}_2^{0,0} + \frac{1}{2}\delta^2\left(\mathcal{K}_2^{0,2} + \mathcal{K}_2^{2,0}\right)\Big\} + \\ &+ \frac{1}{6}\delta^2\left(\mathcal{K}_2^{0,2} - \mathcal{K}_2^{2,0}\right)\cos 2f - \frac{1}{32}\delta^3\left(\mathcal{K}_2^{3,0} - \mathcal{K}_2^{1,2}\right)\cos 3f\Bigg], \end{aligned} \tag{33}$$

$$\begin{aligned} v &= -\delta\sin f + \epsilon\frac{1}{2}\delta\sin f\,\mathcal{K}_1^{1,0} + \\ &+ \frac{\epsilon^2}{2!}\Bigg[\Big\{\frac{1}{4}\delta\left(\left(\mathcal{K}_1^{1,0}\right)^2 + 2\mathcal{K}_2^{1,0}\right) + \frac{1}{8}\delta^3\left(\mathcal{K}_2^{1,2} + 3\mathcal{K}_2^{3,0}\right)\Big\}\sin f \; + \\ &+ \frac{1}{3}\delta^2\left(\mathcal{K}_2^{2,0} - \mathcal{K}_2^{0,2}\right)\sin 2f + \frac{3}{32}\delta^3\left(\mathcal{K}_2^{3,0} - \mathcal{K}_2^{1,2}\right)\sin 3f\Bigg], \end{aligned} \tag{34}$$

where the variation of δ and f with respect to the time s is given by means of the differential equations

$$\frac{\mathrm{d}\delta}{\mathrm{d}s} = 0\,, \tag{35}$$

$$\frac{\mathrm{d}f}{\mathrm{d}s} = n_f$$
$$= 1 - \epsilon\,\frac{\mathcal{K}_1^{1,0}}{2} - \frac{\epsilon^2}{2!}\left(\frac{\mathcal{K}_1^{1,0}\mathcal{K}_1^{1,0}}{4} + \frac{\mathcal{K}_2^{1,0}}{2} + \frac{3\delta^2\mathcal{K}_2^{3,0}}{8} + \frac{\delta^2\mathcal{K}_2^{1,2}}{8}\right). \tag{36}$$

The Equation (35) shows us that δ has a constant value n_f is also a constant since the values of $K_n^{i,j}$ are constants. However, we will not integrate Equation (36) to obtain the relation between f and s, instead of that, we will obtain in the next section a generalized Kepler equation. This equation will give a direct relation between f and t.

Changing to the variables C, S, defined in (28), the Equations (33) and (34) become

$$\begin{aligned} u &= C + \epsilon\mathcal{K}_1^{0,0} + \frac{\epsilon^2}{2!}\left[\left(2\mathcal{K}_1^{0,0}\mathcal{K}_1^{1,0} + \mathcal{K}_2^{0,0}\right) + \frac{C^2}{3}\left(2\mathcal{K}_2^{0,2} + \mathcal{K}_2^{2,0}\right) + \right. \\ &\left. + \frac{S^2}{3}\left(\mathcal{K}_2^{0,2} + 2\mathcal{K}_2^{2,0}\right) + \frac{C^3}{32}\left(\mathcal{K}_2^{1,2} - \mathcal{K}_2^{3,0}\right) - \frac{3CS^2}{32}\left(\mathcal{K}_2^{1,2} - \mathcal{K}_2^{3,0}\right)\right], \\ v &= S - \epsilon\frac{S\mathcal{K}_1^{1,0}}{2} + \frac{\epsilon^2}{2!}\left[-\frac{S}{4}\left(\left(\mathcal{K}_1^{1,0}\right)^2 + 2\mathcal{K}_2^{1,0}\right) + \frac{2SC}{3}\left(\mathcal{K}_2^{0,2} - \mathcal{K}_2^{2,0}\right) - \right. \\ &\left. - \frac{SC^2}{32}\left(21\mathcal{K}_2^{3,0} - 5\mathcal{K}_2^{1,2}\right) - \frac{S^3}{32}\left(9\mathcal{K}_2^{3,0} + 7\mathcal{K}_2^{1,2}\right)\right]. \end{aligned} \tag{37}$$

This equation can be inverted by using the algorithm given in the previous section

$$\begin{aligned} \delta\cos f &= u - \epsilon\mathcal{K}_1^{0,0} + \frac{\epsilon^2}{2!}\left[-\left(2\mathcal{K}_1^{0,0}\mathcal{K}_1^{1,0} + \mathcal{K}_2^{0,0}\right) - \frac{u^2}{3}\left(2\mathcal{K}_2^{0,2} + \mathcal{K}_2^{2,0}\right) - \right. \\ &\left. - \frac{v^2}{3}\left(\mathcal{K}_2^{0,2} + 2\mathcal{K}_2^{2,0}\right) - \frac{u^3}{32}\left(\mathcal{K}_2^{1,2} - \mathcal{K}_2^{3,0}\right) + \frac{3uv^2}{32}\left(\mathcal{K}_2^{1,2} - \mathcal{K}_2^{3,0}\right)\right], \\ -\delta\sin f &= v + \epsilon\frac{v\mathcal{K}_1^{1,0}}{2} + \frac{\epsilon^2}{2!}\left[\frac{v}{4}\left(\left(3\mathcal{K}_1^{1,0}\right)^2 + 2\mathcal{K}_2^{1,0}\right) + \frac{2uv}{3}\left(\mathcal{K}_2^{2,0} - \mathcal{K}_2^{0,2}\right) + \right. \\ &\left. + \frac{vu^2}{32}\left(21\mathcal{K}_2^{3,0} - 5\mathcal{K}_2^{1,2}\right) + \frac{v^3}{32}\left(9\mathcal{K}_2^{3,0} + 7\mathcal{K}_2^{1,2}\right)\right]. \end{aligned} \tag{38}$$

The initial values of δ and f can be obtained by means of (38).

Using the Equations (9), (11) and (6), we may write

$$\frac{1}{r} = u - \beta, \quad R = -\Theta v + \frac{\epsilon^2}{2!}\left[2\mathcal{M}_2^{2,2}\,(u-\beta)^2\,\Theta v\right], \tag{39}$$

and eventually, substituting (33) and (34) in the previous equalities, and applying the formula (18) of the product of power series, we obtain

$$\begin{aligned} R \; = \; & \frac{\mu e}{\Theta}\sin f - \epsilon\mathcal{K}_1^{1,0}\frac{\mu e}{\Theta}\sin f + \\ & + \frac{\epsilon^2}{2!}\Bigg[\Bigg\{-\left(\frac{\mathcal{K}_2^{1,0}}{2} + \frac{(\mathcal{K}_1^{1,0})^2}{4} + \frac{2\mathcal{M}_2^{2,2}}{p^2}\right)\delta\Theta - \\ & - \left(\frac{3\mathcal{K}_2^{3,0}}{8} + \frac{\mathcal{K}_2^{1,2}}{8} + \frac{\mathcal{M}_2^{2,2}}{2}\right)\delta^3\Theta\Bigg\}\sin f + \\ & + \left\{\frac{1}{3}\left(\mathcal{K}_2^{0,2} - \mathcal{K}_2^{2,0}\right) - \frac{2}{p}\mathcal{M}_2^{2,2}\right\}\delta^2\Theta\,\sin 2f + \\ & + \left\{\frac{3}{32}\left(\mathcal{K}_2^{1,2} - \mathcal{K}_2^{3,0}\right) - \frac{1}{2}\mathcal{M}_2^{2,2}\right\}\delta^3\Theta\,\sin 3f\Bigg], \end{aligned} \tag{40}$$

and

$$\begin{aligned} \frac{1}{r} \; = \; & \frac{1 + e\cos f}{p} + \epsilon\mathcal{K}_1^{0,0} + \\ & + \frac{\epsilon^2}{2!}\Bigg[\Bigg\{2\mathcal{K}_1^{0,0}\mathcal{K}_1^{1,0} + \mathcal{K}_2^{0,0} + \frac{\delta^2}{2}\left(\mathcal{K}_2^{0,2} + \mathcal{K}_2^{2,0}\right)\Bigg\} + \\ & + \frac{\delta^2}{6}\left(\mathcal{K}_2^{0,2} - \mathcal{K}_2^{2,0}\right)\cos 2f + \frac{\delta^3}{32}\left(\mathcal{K}_2^{1,2} - \mathcal{K}_2^{3,0}\right)\cos 3f\Bigg], \end{aligned} \tag{41}$$

where we used the generalized semi-latus rectum, eccentricity and semi-mayor axis, defined by the expressions

$$p = \frac{1}{\beta}, \qquad e = \frac{\delta}{\beta}, \qquad a = \frac{p}{1 - e^2}. \tag{42}$$

To obtain r we will make use of the iterative formula

$$c_0 = \frac{1}{a_0}, \qquad c_n = -\frac{1}{a_0}\sum_{i=1}^{n}\binom{n}{i}a_i c_{n-i}, \tag{43}$$

that gives the terms of the inverse power series

$$\sum_{n\geqslant 0}\frac{\epsilon^n}{n!}c_n = \frac{1}{\displaystyle\sum_{n\geqslant 0}\frac{\epsilon^n}{n!}a_n}.$$

Applying this formula to Equation (41) we obtain

$$
\begin{aligned}
r \;=\; & \frac{p}{1+e\cos f} - \epsilon \frac{p^2 \mathcal{K}_1^{0,0}}{(1+e\cos f)^2} + \\
& + \frac{\epsilon^2}{2!}\left[\frac{2p^3 (\mathcal{K}_1^{0,0})^2}{(1+e\cos f)^3} - \right. \\
& - \frac{p^2}{(1+e\cos f)^2} \left\{ \left(2\mathcal{K}_1^{0,0}\mathcal{K}_1^{1,0} + \mathcal{K}_2^{0,0} + \frac{\delta^2}{2}\left(\mathcal{K}_2^{0,2} + \mathcal{K}_2^{2,0}\right)\right) + \right. \\
& \left.\left. + \frac{\delta^2}{6}\left(\mathcal{K}_2^{0,2} - \mathcal{K}_2^{2,0}\right)\cos 2f + \frac{\delta^3}{32}\left(\mathcal{K}_2^{1,2} - \mathcal{K}_2^{3,0}\right)\cos 3f \right\}\right]. \qquad (44)
\end{aligned}
$$

6. Integration of the Rest of the Variables

Taking into account the relation

$$
\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\Theta}{r^2} n_f \frac{\mathrm{d}}{\mathrm{d}f},
$$

the Hamilton equations corresponding to the variables θ, ν become

$$
n_f \frac{\mathrm{d}\theta}{\mathrm{d}f} = \frac{r^2}{\Theta}\frac{\partial \mathcal{H}}{\partial \Theta}, \qquad n_f \frac{\mathrm{d}\nu}{\mathrm{d}f} = \frac{r^2}{\Theta}\frac{\partial \mathcal{H}}{\partial N}, \qquad (45)
$$

and computing the partial derivative of $\mathcal{H}$ with respect to the momenta in (3), we obtain

$$
\begin{aligned}
n_f \frac{\mathrm{d}\theta}{\mathrm{d}f} \;&=\; 1 + \epsilon \mathcal{M}_{\Theta 1}^{0,2} + \frac{\epsilon^2}{2!}\left(\mathcal{M}_{\Theta 2}^{2,2} R^2 + \mathcal{M}_{\Theta 2}^{0,2} + \mathcal{M}_{\Theta 2}^{0,3}\frac{1}{r} + \mathcal{M}_{\Theta 2}^{0,4}\frac{1}{r^2}\right), \\
n_f \frac{\mathrm{d}\nu}{\mathrm{d}f} \;&=\; \epsilon \mathcal{M}_{N1}^{0,2} + \frac{\epsilon^2}{2!}\left(\mathcal{M}_{N2}^{2,2} R^2 + \mathcal{M}_{N2}^{0,2} + \mathcal{M}_{N2}^{0,3}\frac{1}{r} + \mathcal{M}_{N2}^{0,4}\frac{1}{r^2}\right), \qquad (46)
\end{aligned}
$$

where

$$
\mathcal{M}_{\Theta n}^{i,j} = \frac{1}{\Theta}\frac{\partial \mathcal{M}_n^{i,j}}{\partial \Theta}, \qquad \mathcal{M}_{Nn}^{i,j} = \frac{1}{\Theta}\frac{\partial \mathcal{M}_n^{i,j}}{\partial N}.
$$

Eventually, substituting R, $1/r$ by their expressions (40) and (41) in terms of f and integrating (46), we have

$$
n_f(\theta - T_\theta) \;=\; f + \epsilon \mathcal{M}_{\Theta 1}^{0,2} f + \frac{\epsilon^2}{2!}\left[\left(\mathcal{M}_{\Theta 2}^{0,2} + \frac{\mathcal{M}_{\Theta 2}^{0,3}}{p} + \frac{\mathcal{M}_{\Theta 2}^{0,4}}{p^2} + \frac{\delta^2 \mathcal{M}_{\Theta 2}^{0,4}}{2} + \right.\right.
$$

$$+\frac{\delta^2\Theta^2\mathcal{M}^{2,2}_{\Theta 2}}{2}\Bigg) f + \left(\delta\mathcal{M}^{0,3}_{\Theta 2} + \frac{2\delta\mathcal{M}^{0,4}_{\Theta 2}}{p}\right)\sin f \times$$

$$\times\left(\frac{\delta^2\mathcal{M}^{0,4}_{\Theta 2}}{4} - \frac{\delta^2\Theta^2\mathcal{M}^{2,2}_{\Theta 2}}{4}\right)\sin 2f\Bigg],$$

$$n_f(\nu - T_\nu) = \epsilon\mathcal{M}^{0,2}_{N1} f + \frac{\epsilon^2}{2!}\Bigg[\Bigg(\mathcal{M}^{0,2}_{N2} + \frac{\mathcal{M}^{0,3}_{N2}}{p} + \frac{\mathcal{M}^{0,4}_{N2}}{p^2} + \frac{\delta^2\mathcal{M}^{0,4}_{N2}}{2} +$$

$$+\frac{\delta^2\Theta^2\mathcal{M}^{2,2}_{N2}}{2}\Bigg) f + \left(\delta\mathcal{M}^{0,3}_{N2} + \frac{2\delta\mathcal{M}^{0,4}_{N2}}{p}\right)\sin f \times$$

$$\times\left(\frac{\delta^2\mathcal{M}^{0,4}_{N2}}{4} - \frac{\delta^2\Theta^2\mathcal{M}^{2,2}_{N2}}{4}\right)\sin 2f\Bigg], \tag{47}$$

where T_θ, T_ν are the value of θ, ν when $f = 0$.

7. Generalized Kepler Equation

The variation of the generalized true anomaly with respect to the time will be obtained after considering the definition of s given by (9) from which we have

$$n_f\,\Theta\,\mathrm{d}t = r^2\,\mathrm{d}f. \tag{48}$$

Substituting (44) into (48) we have

$$n_f\,\Theta\,\mathrm{d}t = \frac{p^2\,\mathrm{d}f}{(1+e\cos f)^2} - \epsilon\frac{2p^3\mathcal{K}^{0,0}_1\,\mathrm{d}f}{(1+e\cos f)^3} + \frac{\epsilon^2}{2!}\Bigg[\frac{6p^4(\mathcal{K}^{0,0}_1)^2}{(1+e\cos f)^4} -$$

$$-\frac{2p^3}{(1+e\cos f)^3}\Bigg\{\left(2\mathcal{K}^{0,0}_1\mathcal{K}^{1,0}_1 + \mathcal{K}^{0,0}_2 + \frac{\delta^2}{2}\left(\mathcal{K}^{0,2}_2 + \mathcal{K}^{2,0}_2\right)\right) +$$

$$+\frac{\delta^2}{6}\left(\mathcal{K}^{0,2}_2 - \mathcal{K}^{2,0}_2\right)\cos 2f + \frac{\delta^3}{32}\left(\mathcal{K}^{1,2}_2 - \mathcal{K}^{3,0}_2\right)\cos 3f\Bigg\}\Bigg]\,\mathrm{d}f. \tag{49}$$

We apply, now, a change identical to the transformation from the true anomaly to the eccentric anomaly in the non-perturbed two body problem

$$\cos f = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}, \qquad \sin f = \frac{\cos E - e}{1-e\cos E}, \tag{50}$$

in which f and E represent now the generalized anomalies. From that change we easily obtain

$$\frac{p}{1+e\cos f} = a(1-e\cos E), \qquad \mathrm{d}f = \frac{\sqrt{1-e^2}}{1-e\cos E}\,\mathrm{d}E. \tag{51}$$

This change of variable is usually used in orbital mechanics to transform integrals with powers of $(1 + e\cos f)$ in the denominator into terms with powers of $(1 - e\cos E)$ in the numerator: in this way we will be able to integrate these expressions. In our problem, the appearance of factors $\cos nf$, $\sin nf$ in the numerator of the perturbation terms produces that even after applying the change, some terms will still have powers of $(1 - e\cos E)$ in the denominator.

A practical way to avoid this formal complication consists in applying firstly the change from f to w defined by

$$w = 1 + e\cos f, \qquad \cos f = \frac{w-1}{e}.$$

Using the property (30) we obtain

$$\cos nf = T_n\left(\frac{w-1}{e}\right), \qquad \sin nf = \sin f\, U_{n-1}\left(\frac{w-1}{e}\right), \tag{52}$$

that permit to express every term of (49) by positive or negative powers of w, multiplied or not by $\sin f$. Then, to integrate (49) is sufficient to consider the following integrals:

$$\int w^n\,\mathrm{d}f = \begin{cases} \displaystyle\int (1 + e\cos f)^n\,\mathrm{d}f, & n \geqslant 0, \\ \displaystyle\eta^{-2n-1}\int (1 - e\cos E)^{-n-1}\,\mathrm{d}E, & n < 0, \end{cases}$$

$$\int \sin f\, w^n\,\mathrm{d}f = \begin{cases} \displaystyle\int \sin f\,(1 + e\cos f)^n\,\mathrm{d}f, & n \geqslant 0, \\ \displaystyle -\frac{1}{e}\ln(1 + e\cos f), & n = -1, \\ \displaystyle\eta^{-2n-2}\int \sin E\,(1 - e\cos E)^{-n-2}\,\mathrm{d}E, & n < -1, \end{cases} \tag{53}$$

with $\eta = \sqrt{1 - e^2}$.

In particular, after multiplying by η^3/p^2, for the second order we have

$$\begin{aligned} n\,\mathrm{d}t &= \eta^3\frac{\mathrm{d}f}{w^2} + \epsilon\eta^3\mathcal{W}_1^{-3}\frac{\mathrm{d}f}{w^3} + \frac{\epsilon^2}{2!}\eta^3 \times \\ &\quad \times \left(\mathcal{W}_2^0\mathrm{d}f + \mathcal{W}_2^{-1}\frac{\mathrm{d}f}{w} + \mathcal{W}_2^{-2}\frac{\mathrm{d}f}{w^2} + \mathcal{W}_2^{-3}\frac{\mathrm{d}f}{w^3} + \mathcal{W}_2^{-4}\frac{\mathrm{d}f}{w^4}\right), \end{aligned} \tag{54}$$

where $n = n_f\sqrt{\mu/a^3}$ is the generalized mean motion, and

$$\begin{aligned} \mathcal{W}_1^{-3} &= -2p\mathcal{K}_1^{0,0}, \\ \mathcal{W}_2^0 &= \frac{1}{4p^2}\left(\mathcal{K}_2^{3,0} - \mathcal{K}_2^{1,2}\right), \end{aligned}$$

$$\begin{aligned}
\mathcal{W}_2^{-1} &= \frac{3}{4p^2}\left(\mathcal{K}_2^{1,2}-\mathcal{K}_2^{3,0}\right)+\frac{2}{3p}\left(\mathcal{K}_2^{2,0}-\mathcal{K}_2^{0,2}\right),\\
\mathcal{W}_2^{-2} &= \frac{3}{4p^2}\left(\mathcal{K}_2^{0,3}-\mathcal{K}_2^{1,2}\right)+\frac{4}{3p}\left(\mathcal{K}_2^{0,2}-\mathcal{K}_2^{2,0}\right)+\frac{3e^2}{16p^2}\left(\mathcal{K}_2^{1,2}-\mathcal{K}_2^{3,0}\right),\\
\mathcal{W}_2^{-3} &= \frac{1}{4p^2}\left(\mathcal{K}_2^{1,2}-\mathcal{K}_2^{3,0}\right)-2p\left(2\mathcal{K}_1^{0,0}\mathcal{K}_1^{1,0}-\mathcal{K}_2^{0,0}\right)\\
&\quad+\frac{2}{3p}\left(\mathcal{K}_2^{2,0}-\mathcal{K}_2^{0,2}\right)-\frac{2e^2}{3p}\left(2\mathcal{K}_2^{2,0}+\mathcal{K}_2^{0,2}\right)+\frac{3e^2}{16p^2}\left(\mathcal{K}_2^{3,0}-\mathcal{K}_2^{1,2}\right),\\
\mathcal{W}_2^{-4} &= 6p^2\left(\mathcal{K}_1^{0,0}\right)^2.
\end{aligned}$$

Finally, integrating (54), we obtain

$$\begin{aligned}
n(t-T) &= E - e\sin E + \epsilon\left[\frac{(2+e^2)\mathcal{W}_1^{-3}}{2\eta^2}E - \frac{2e\mathcal{W}_1^{-3}}{\eta^2}\sin E +\right.\\
&\quad\left.+\frac{e^2\mathcal{W}_1^{-3}}{4\eta^2}\sin 2E\right]+\frac{\epsilon^2}{2!}\left[\eta^3\mathcal{W}_2^0 f + \left(\eta^2\mathcal{W}_2^{-1}+W_2^{-2}+\right.\right.\\
&\quad\left.+\frac{(2+e^2)\mathcal{W}_2^{-3}}{2\eta^2}+\frac{(2+3e^2)\mathcal{W}_2^{-4}}{2\eta^4}\right)E-\\
&\quad-e\left(\mathcal{W}_2^{-2}+\frac{2\mathcal{W}_2^{-3}}{\eta^2}+\frac{3(4+e^2)\mathcal{W}_2^{-4}}{\eta^4}\right)\sin E\times\\
&\quad\times\left.\left(\frac{e^2\mathcal{W}_2^{-3}}{4\eta^2}+\frac{3e^2\mathcal{W}_2^{-4}}{4\eta^4}\right)\sin 2E-\frac{e^3\mathcal{W}_2^{-4}}{12\eta^4}\sin 3E\right], \qquad (55)
\end{aligned}$$

where T represents the value of t when $f = E = 0$.

Note that for orders greater than one, terms in f and E appear mixed in the generalized Kepler equation.

8. Numerical Tests

In the graphics 1 and 2, we compare the integrals obtained by applying the KBM–method to the Hamiltonian (2) until third order (Figure 1) and fourth order (Figure 2) versus the numerical integration of (2) when using a Runge-Kutta method of 8th order and 13 stages. The programs of evaluation of ephemeris from the analytical theory are written automatically by ATESAT.

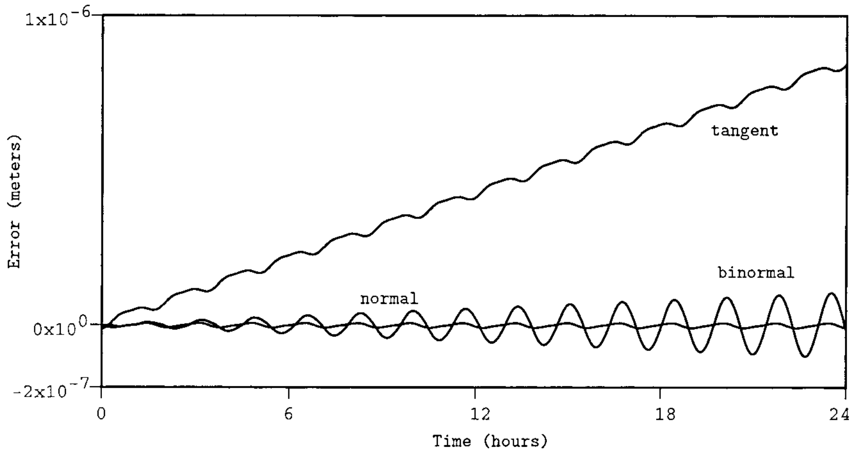

Figure 1. Analytical theory of order 3 versus numerical integration.

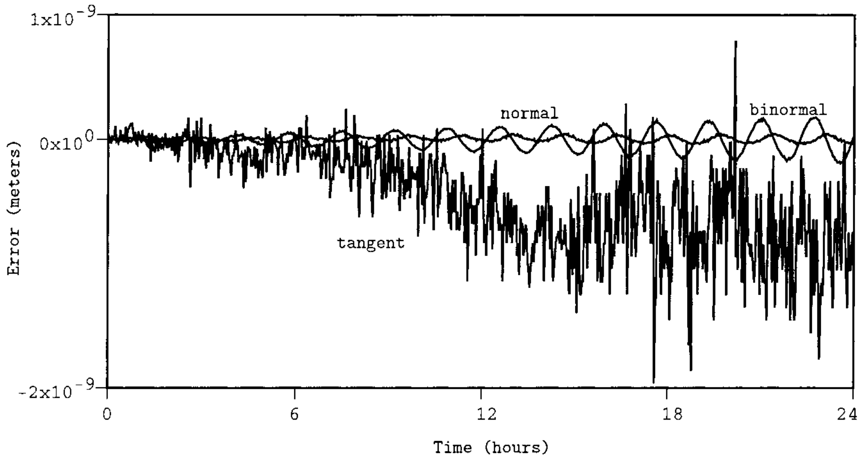

Figure 2. Analytical theory of order 4 versus numerical integration.

The graphics show the variations of the position in tangent, normal and binormal direction of a satellite with the following initial conditions:

$$a_0 = 7834.999\,\text{kms}, \qquad e_0 = 0.0009999, \qquad i_0 = 55^o.$$

In Figure 3 we show a comparison between a third order theory in which the Delaunay normalization has been used instead of the KBM versus the same numerical method. The same initial conditions have been used. The KBM method presents, in this particular case, with a very small eccentricity, better numerical results. More details about the differences between Delaunay normalization and KBM–method are showed by San–Juan (1998).

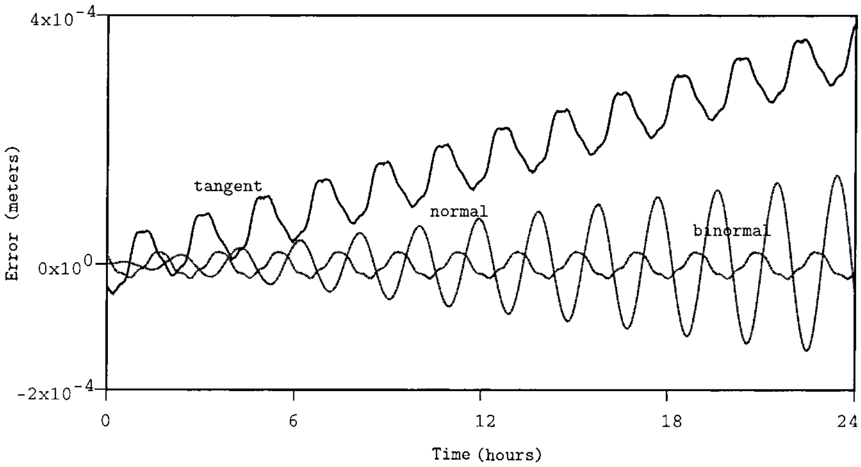

Figure 3. Analytical theory of order 3 (Delaunay normalization instead KBM method) versus numerical integration.

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Appendix: Inversion of Series by Using Non-canonical Lie Transformations

DEFINITION. We call non-canonical Lie transformation of generator $\mathbf{W}$ to the solution

$$\varphi\colon\ (\mathbf{y}, \epsilon) \to \mathbf{x}(\mathbf{y}, \epsilon)\colon\ \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \tag{56}$$

of the ordinary differential equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\epsilon} = \mathbf{W}(\mathbf{x}; \epsilon) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!}\mathbf{W}_{n+1}(\mathbf{x}), \quad \mathbf{x}(\mathbf{y}, 0) = \mathbf{y}. \tag{57}$$

To apply a Lie transformation (57) to the function

$$\mathbf{F}(\mathbf{x}; \epsilon) = \sum_{n\geqslant 0} \frac{\epsilon^n}{n!}\mathbf{F}_{n,0}(\mathbf{x}), \tag{58}$$

we will use the Lie triangle

$$\mathbf{F}_{n,p} = \mathbf{F}_{n+1,p-1} + \sum_{i+j=n} \frac{n!}{i!j!} \left(\nabla_{\mathbf{x}} \mathbf{F}_{i,p-1} \cdot \mathbf{W}_{j+1} \right). \tag{59}$$

that gives an iterative way to obtain the terms $\mathbf{F}_{0,n}$ of the transformed function

$$\varphi^* \mathbf{F}(\mathbf{y};\epsilon) = \mathbf{F}(\mathbf{x}(\mathbf{y},\epsilon);\epsilon) = \sum_{n \geqslant 0} \frac{\epsilon^n}{n!} \mathbf{F}_{0,n}(\mathbf{y}). \tag{60}$$

Applying the transformation to the function $\mathbf{F}(\mathbf{x},\epsilon) = \mathbf{x}$, we obtain the explicit solution

$$\mathbf{x} = \sum_{n \geqslant 0} \frac{\epsilon^n}{n!} \mathbf{x}_n(\mathbf{y}), \quad \mathbf{x}_0(\mathbf{y}) = \mathbf{y}, \tag{61}$$

of the non-canonical Lie transformation (57).

PROPOSITION. The inverse of a non-canonical Lie transformation (57) whose explicit expression is given by (61) is a non-canonical Lie transformation whose generator $\mathbf{V}(\mathbf{y};\epsilon)$ is given, order by order, by the expressions

$$\begin{aligned} \mathbf{V}_1 &= -\mathbf{x}_1, \\ \mathbf{V}_{n+1} &= -\mathbf{x}_{n+1} - \sum_{i=0}^{n-1} \binom{n}{i} (\nabla_y \mathbf{x}_j \cdot \mathbf{V}_{i+1}), \quad n \geqslant 1. \end{aligned} \tag{62}$$

This result may be applied to obtain the expression

$$\mathbf{y} = \sum_{n \geqslant 0} \frac{\epsilon^n}{n!} \mathbf{y}_n(\mathbf{x}), \qquad \mathbf{y}_0(\mathbf{x}) = \mathbf{x}, \tag{63}$$

of the inverse of the series (61).

A detailed description of the properties and applications of these transformations can be found in Kamel (1970), Henrard (1970) and Deprit (1969). In Deprit (1979) we find the proof of the proposition.

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