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DIFFERENTIATION OF INTEGRALS IN $\mathbf{R}^\omega$

J. M. ALDAZ (La Rioja)

Abstract. We show that the Lebesgue differentiation theorem does not hold in $\mathbf{R}^\omega$ with the "product" Lebesgue measure.

1. Introduction

The validity of the Lebesgue differentiation theorem beyond the setting of $\mathbf{R}^n$ has been investigated by several authors. In the case of infinite dimensional separable Hilbert spaces, since there is no analogue of Lebesgue measure, these studies have usually considered gaussian measures instead. D. Preiss proved that there exist gaussian measures for which the result fails (cf. [3] and [4]), while J. Tišer showed in [5] that for certain gaussian measures the Lebesgue differentiation theorem does hold. In [1], R. Baker constructed a version λ^ω of Lebesgue measure in the separable Frechet space $\mathbf{R}^\omega$, such that λ^ω is a translation invariant Borel measure which assigns to each rectangle $\Pi_n(a_n, b_n)$ its volume $\Pi_n(b_n - a_n)$ (whenever the product exists and is finite). A natural question to ask is how the differentiation of integrals behaves in this setting. We shall see that the Lebesgue differentiation theorem fails in a somewhat surprising way: If $f \in L^1(\mathbf{R}^\omega, \lambda^\omega)$, then for almost every point with respect to λ^ω,

$$\lim \frac{1}{\lambda^\omega(C)} \int_{x+C} f \, d\lambda^\omega = 0,$$

where C is a rectangle centered at zero with $0 < \lambda^\omega(C) < \infty$, and taking the limit simply means that $\operatorname{diam}(C) \to 0$. The analogous result also holds for the corresponding maximal operator.

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2. Preliminary definitions and results

2.1. Definition. By a *rectangle* in $\mathbf{R}^\omega$ we mean a set of the form $\Pi_{n=0}^{\infty} I_n$, where each I_n is an interval, which can be of infinite length, or degenerate, such as $[a,a]$ or (a,a).

The basic result (see Theorem I of [1]) we need on the construction of λ^ω follows next. Let $\mathcal{R}$ be the class of all infinite dimensional rectangles $R \subset \mathbf{R}^\omega := \Pi_{n=0}^{\infty}\mathbf{R}$ of the form $R = \Pi_{n=0}^{\infty}(a_n, b_n)$, such that $-\infty < a_n \leqq b_n < \infty$ and $0 \leqq \Pi_{n=0}^{\infty}(b_n - a_n) < \infty$. Define τ on $\mathcal{R}$ by setting $\tau(R) = \Pi_{n=0}^{\infty}(b_n - a_n)$.

2.2. Theorem. *Let E be a subset of $\mathbf{R}^\omega$, and let the set function λ^ω be given by $\lambda^\omega(E) := \inf\left\{ \sum_{n=0}^{\infty}\tau(R_n)\text{: for every } n,\ R_n \in \mathcal{R} \text{ and } E \subset \cup_{n=0}^{\infty} R_n \right\}$, with the convention that $\inf\emptyset = \infty$. Then λ^ω is a translation invariant Borel measure on $\mathbf{R}^\omega$, satisfying $\lambda^\omega\big(\Pi_{n=0}^{\infty}(a_n,b_n)\big) = \Pi_{n=0}^{\infty}(b_n - a_n)$ for every $\Pi_{n=0}^{\infty}(a_n,b_n) \in \mathcal{R}$.*

2.3. Definitions. Let T be a subadditive operator with domain a linear space of measurable functions on (X,μ), and taking measurable functions (possibly from another space) as values. Then T is of *weak type* (p,q), where $p,q \in [1,\infty)$, if there exists a constant c such that for every $f \in \operatorname{Dom} T$ with $f \in L^p(\mu)$, and every $\alpha > 0$, $\alpha^q \mu\big(\{|Tf| > \alpha\}\big) \leqq \big(c\|f\|_p\big)^q$. A rectangle R is *admissible* if $0 < \lambda^\omega(R) < \infty$. We use $\mathcal{A}$ to denote the class of admissible rectangles.

3. Admissible rectangles

In this section we present some results about the measure of rectangles. Denote by π_n the projection from $\mathbf{R}^\omega$ onto the n-th coordinate. Given a sequence $\{u_n\}$ of real numbers, to say that the product $\Pi_n u_n$ is convergent only means that the limit $\lim_n \Pi_0^n u_k$ exists. However, this definition gives a very special role to zero (one zero suffices to make the product converge, regardless of what the other factors do). To remove this inconvenience, quite often a more restrictive definition is used: The infinite product $\Pi_n u_n$ is convergent if there exists an $m \in \omega$ such that the sequence of products $\left\{\Pi_{k=0}^{n} u_{m+k}\right\}_{n=0}^{\infty}$ converges to a strictly positive real number. With this definition, a necessary condition for convergence is that $\lim_n u_n = 1$. Note that if the limit exists in the first sense, and $0 < \Pi_n u_n$, then the product is convergent in the stricter sense also. It follows that if $\Pi_n I_n \in \mathcal{R} \cap \mathcal{A}$, then $\lim_n |I_n| = 1$. So we have

3.1. Lemma. *Let $R = \Pi_n I_n \in \mathcal{R}$. If $\tau(R) > 0$, then $\lim_n |I_n| = 1$.*

It is to be expected that if we shrink an infinite number of sides of $\Pi_n I_n \in \mathcal{R}$, by at least a fixed positive amount, then the new rectangle will have measure zero. This is the content of the next lemma.

3.2. Lemma. *Let $R = \Pi_n I_n \in \mathcal{R}$, and let A be an infinite subset of ω. Suppose there exists an $a > 0$ such that for every $n \in A$, $a \leqq |I_n|$. If $n \in A$, let $J_n \subset I_n$ be an interval with $|J_n| \leqq |I_n| - a$, and if $n \in A^c$, set $J_n := I_n$. Then $\tau(\Pi_n J_n) = 0$.*

Proof. Suppose $\tau(R) > 0$. Let $a_k = |I_k| - |J_k|$. Then

$$\frac{\Pi_{k=0}^n |J_k|}{\Pi_{k=0}^n |I_k|} = \Pi_{k=0}^n \left(1 - \frac{a_k}{|I_k|}\right).$$

Since $\limsup \frac{a_n}{|I_n|} \geqq a$, letting n go to infinity we get that $\lim \Pi_{k=0}^n |J_k| = 0$. $\square$

Next we show that no admissible rectangle is degenerate.

3.3. Proposition. *Let $R = \Pi_n I_n$ be a rectangle with $\lambda^\omega(R) < \infty$. If there exists an m such that $|I_m| = 0$, then $\lambda^\omega(R) = 0$.*

Proof. Let $\{R_j\}$ be a sequence of rectangles such that $R_j \in \mathcal{R}$ for every $j \in \omega$, $R \subset \cup_{j=0}^\infty R_j$, and $\sum_{j=0}^\infty \tau(R_j) = b < \infty$. We may, without loss of generality, assume that for every j, $R \cap R_j \neq \emptyset$. Fix a positive natural number N. Choose m so that $|I_m| = 0$, and for each $j \in \omega$, let R_j' be obtained from R_j by replacing $\pi_m R_j$ with an open interval H^j such that $I_m \subset H^j \subset \pi_m R_j$ and $|H^j| \leqq N^{-1} |\pi_m R_j|$. Since $R \subset \cup_{j=0}^\infty R_j'$, $\lambda^\omega(R) \leqq N^{-1} b$ and the result follows by letting $N \to \infty$. $\square$

Lemma 3.1 tells us that for every admissible rectangle $\Pi_n I_n$ from $\mathcal{R}$, $\lim_n |I_n| = 1$. In fact, every admissible rectangle has this property.

3.4. Proposition. *Let $R = \Pi_n I_n$ be an admissible rectangle. Then $\lim |I_n| = 1$.*

Proof. First we show that $\liminf_n |I_n| \geqq 1$. If $\liminf_n |I_n| < 1$, then there exist $a < b < 1$ and a subsequence $\{n_i\}$ such that for every $i \in \omega$, we have $|I_{n_i}| < a$. Let $a_n < b_n$ be the extreme points of I_n, and let $\{R_j\}$ be a sequence of rectangles from $\mathcal{R}$ such that $R \subset \cup_{j=0}^\infty R_j$ and $\sum_{j=0}^\infty \tau(R_j) < \infty$. Let B be the rectangle defined by $\pi_n B = \mathbf{R}$ if $n \notin \{n_i : i \in \omega\}$, and $\pi_n B = \left(a_n - \frac{b-a}{2}, b_n + \frac{b-a}{2}\right)$ otherwise. Since $b_n + \frac{b-a}{2} - \left(a_n - \frac{b-a}{2}\right) < b$, it follows from Lemma 3.2 that for each j, $R_j' := R_j \cap B$ is a rectangle with $\tau(R_j') = 0$. But $R \subset \cup_{j=0}^\infty R_j'$, so $\lambda^\omega(R) = 0$, contradicting the fact that R is admissible.

Suppose next that $\limsup_n |I_n| > 1$. By translation invariance we may assume that R is centered at zero, and as before, we choose a sequence $\{R_j\}$ of rectangles from $\mathcal{R}$ with $R \subset \cup_{j=0}^\infty R_j$ and $\sum_{j=0}^\infty \tau(R_j) < \infty$. Let $a > 0$ be such that for some infinite subset A of ω, $|I_n| > 1 + a$ whenever $n \in A$. Now there exists an $m \in \omega$ with $\lambda^\omega(R \cap R_m) > 0$ (otherwise R would have measure zero). Define $B := R \cap R_m$, and note that $\limsup_n \left|\pi_n(B)\right| \leqq 1$, by Lemma 3.1. Let z be the center of B, and let $C = -z + B$. Then $C \subset R$, since both rectangles are centered at zero, and every side of C has length less than or equal to the corresponding side of R. Pick $N \in \omega$ with $|\pi_n C| \leqq 1 + a/2$ whenever $n \geqq N$, and define $T_n : \mathbf{R} \to \mathbf{R}$ by setting $T_n(x) = \left(\frac{2+a}{2+2a}\right) x$ if $n \in A \setminus \{k < N\}$, and $T_n(x) = x$ otherwise. Let T be the linear transformation defined on $\mathbf{R}^\omega$ by (T_n). Then $T(R)$ is a rectangle centered at zero which satisfies $C \subset T(R) \subset R$, since for every $n \in A \setminus \{k < N\}$,

$$|\pi_n C| \leqq 1 + a/2 \leqq \left|\pi_n T(R)\right| \leqq |\pi_n R|.$$

Note next that if $\lambda^\omega(R_j) = 0$, then $\lambda^\omega\big(T(R_j)\big) = 0$, while if $\lambda^\omega(R_j) > 0$, then T shrinks an infinite number of sides of R_j by a fixed, positive amount, since $\lim_n |\pi_n R_j| = 1$. So by Lemma 3.2, we also have that $\lambda^\omega\big(T(R_j)\big) = 0$. But $\big\{T(R_j)\big\}$ is a cover of $T(R)$, whence

$$0 < \lambda^\omega(R \cap R_m) = \lambda^\omega(C) \leqq \lambda^\omega\big(\cup_j T(R_j)\big) = 0. \quad \square$$

4. Behaviour of the Hardy–Littlewood maximal function, and differentiation of integrals

Given a collection of rectangles $\mathcal{Q}$, the symbol $\mathcal{Q}_0$ stands for those rectangles in $\mathcal{Q}$ which are centered at zero. Let $\mathcal{C}$ denote the family of admissible cubes, that is, the cubes of side length one. Note that the collection of admissible cubes is very small (and unsuitable to define differentiation of integrals). Nevertheless, the uncentered Hardy–Littlewood maximal operator

$$M^u_{\mathcal{C}} f(x) := \sup_{\{C \in \mathcal{C} \,:\, x \in C\}} \frac{1}{\lambda^\omega(C)} \int_C |f| d\lambda^\omega$$

associated to $\mathcal{C}$, satisfies no weak bounds.

4.1. Proposition. *For every pair $p, q \in [1, \infty)$, the operator $M^u_{\mathcal{C}}$ does not satisfy any weak type (p, q) inequality.*

Proof. We show that $\lambda^\omega\big(\{M^u_{\mathcal{C}} \chi_{[0,1]^\omega} \geqq 1/2\}\big) = \infty$. Since $\chi_{[0,1]^\omega} \in L^p(\mathbf{R}^\omega, \lambda^\omega)$ for every $p \in [1, \infty)$, this entails the result. For each $n \in \omega$, let

$$U_n = \Pi_0^{n-1}(0,1) \times (1/2, 3/2) \times \Pi_{n+1}^\infty(0,1).$$

Then $\lambda^\omega\big(U_n \setminus [0,1]^\omega\big) = 1/2$, and for $i \neq j$, $\big(U_i \setminus [0,1]^\omega\big) \cap \big(U_j \setminus [0,1]^\omega\big) = \emptyset$, whence $\lambda^\omega(\cup_n U_n) = \infty$. But $\cup_n U_n \subset \big\{ M^u_{\mathcal{C}} \chi_{[0,1]^\omega} \geqq 1/2 \big\}$. $\square$

However, for centered admissible rectangles the situation changes radically (see Corollary 4.6): The corresponding maximal operator vanishes on $L^1(\mathbf{R}^\omega, \lambda^\omega)$. This is quite surprising, not only by comparison with the previous result, but also by considering the analogous situation in $\mathbf{R}^n$: If $n \geqq 2$, then the Hardy–Littlewood maximal operator associated to the centered rectangles with sides parallel to the axes does not satisfy any weak type (1,1) inequality (see [2]).

4.2. DEFINITION. For $a \geqq 0$ and $x \in \mathbf{R}^\omega$, the *a-star* of x, denoted by S^a_x, is the set $S^a_x := \big\{ (y_n) \in \mathbf{R}^\omega : y_n \in [x_n - a, x_n + a]$ save for a finite number of exceptions$\big\}$.

4.3. LEMMA. *For every $a \in (0, 1/2)$ and all $x \in \mathbf{R}^\omega$, the a-star of x has λ^ω-measure zero.*

PROOF. Fix $a \in (0, 1/2)$, and select $b \in (a, 1/2)$. For each finite subset $\{n_0, \dots, n_k\} \subset \omega$, and every $j \in \omega$, write $I^j_{n_i} := (-j, j)$. Then the rectangle

$$R^j_{\{n_1,\dots,n_k\}} := \Pi^k_{i=0} I^j_{n_i} \times \Pi_{\omega \setminus \{n_1,\dots,n_k\}} (x_i - b, x_i + b)$$

belongs to $\mathcal{R}$ and has measure zero. Therefore

$$S^a_{\{n_1,\dots,n_k\}} := \cup_j \Pi^k_{i=0} I^j_{n_i} \times \Pi_{\omega \setminus \{n_1,\dots,n_k\}} [x_i - a, x_i + a] \subset \cup_j R^j_{\{n_1,\dots,n_k\}}$$

also has measure zero. Since the collection of finite subsets of ω is countable, and $S^a_x = \cup_{\{n_1,\dots,n_k\}} S^a_{\{n_1,\dots,n_k\}}$, where the union is taken over all finite subsets of ω, it follows that $\lambda^\omega(S^a_x) = 0$. $\square$

4.4. LEMMA. *Let $a \in (0, 1/2)$, let R be a rectangle from $\mathcal{R}$ with center z, and let C be an admissible rectangle centered at zero. If $x \notin S^a_z$, then $\lambda^\omega\big((x + C) \cap R\big) = 0$.*

PROOF. Fix $R \in \mathcal{R}$. Let $z = (z_n)$ be the center of R, let $x = (x_n) \notin S^a_z$, and let C be an admissible rectangle with center zero. We may assume that $\tau(R) > 0$. Write $R = \Pi_n I_n$, $C = \Pi_n J_n$, and for every $n \in \omega$ define $L_n := I_n \cap (x_n + J_n)$. Let A be an infinite subset of ω such that for every $n \in A$, $|z_n - x_n| > a$. Since $\lim |I_n| = \lim |J_n| = 1$ (Proposition 3.4), there exists an $N \in \omega$ such that for every $n \geqq N$, we have $|I_n| \leqq 1 + a/2$ and $|J_n| \leqq 1 + a/2$. It follows that for every $n \in A \setminus \{k < N\}$, $|L_n| < 1 + a/2 - a = 1 - a/2$, whence $(x + C) \cap R$ has λ^ω-measure zero (Lemma 3.2). $\square$

4.5. THEOREM. *If $f \in L^1(\mathbf{R}^\omega, \lambda^\omega)$, then for every admissible rectangle C centered at zero, and for almost every $x \in \mathbf{R}^\omega$, $\int_{x+C} |f| d\lambda^\omega = 0$.*

Proof. Let $f \in L^1(\mathbf{R}^\omega, \lambda^\omega)$. For each positive rational number r the set $\{|f| \geqq r\}$ has finite measure, so there is a countable cover $\{R_n^r\}$ by rectangles from $\mathcal{R}$ with $\sum_n \lambda^\omega(R_n^r) < \infty$. Thus, $\{f \neq 0\}$ can be covered by a countable collection of rectangles $\{E_n\}$ with $E_n \in \mathcal{R}$ and $\lambda^\omega(E_n) < \infty$ for all $n \in \omega$. Let $a \in (0, 1/2)$, and let e_n be the center of E_n. Then $\lambda^\omega(\cup_n S_{e_n}^a) = 0$ (Lemma 4.3), so for almost all $x \in \mathbf{R}^\omega$, every $C \in \mathcal{A}_0$, and every $n \in \omega$, $\lambda^\omega\big((x + C) \cap E_n\big) = 0$ (Lemma 4.4). Therefore

$$\lambda^\omega\big((x + C) \cap \cup_n E_n\big) = \lambda^\omega\Big(\cup_n \big((x + C) \cap E_n\big)\Big) = 0$$

and $\int_{x+C} |f| d\lambda^\omega = 0$ for λ^ω-almost every x. $\square$

The centered Hardy–Littlewood maximal operator associated to the family $\mathcal{A}$ of admissible rectangles, is defined by

$$M_{\mathcal{A}} f(x) := \sup_{\{C \in \mathcal{A}_0\}} \frac{1}{\lambda^\omega(C)} \int_{x+C} |f| d\lambda^\omega.$$

4.6. Corollary. *For every $f \in L^1(\mathbf{R}^\omega, \lambda^\omega)$ and almost every x, we have $M_{\mathcal{A}} f(x) = 0$.*

Let $\mathcal{C}_0$ be the collection of bounded cubes in $\mathbf{R}^n$ with sides parallel to the axes and centered at 0, directed by reverse inclusion (so $C_1 \geqq C_2$ iff $C_1 \subset C_2$), and let $\mathcal{A}_0$ be the corresponding collection of rectangles. The Lebesgue differentiation theorem (for $\mathbf{R}^n$) with respect to centered cubes tells us that for every $f \in L^1(\mathbf{R}^n)$ and almost every x, $\lim_{\mathcal{C}_0} \frac{1}{|C|} \int_{x+C} f = f(x)$. It is well known, however, that if we replace $\mathcal{C}_0$ by $\mathcal{A}_0$, then there exists a function $f \in L^1(\mathbf{R}^2)$ for which $\lim_{\mathcal{A}_0} \frac{1}{|C|} \int_{x+C} f$ fails to exist on a set of positive measure (see [2]). In $\mathbf{R}^\omega$ the Lebesgue differentiation theorem with respect to centered rectangles also fails but for a different reason: the limit exists and is zero almost always for every integrable function (Corollary 4.7). Note that the admissible rectangles centered at a point x are not directed by reverse inclusion: the intersection of two rectangles of positive measure with center x is a rectangle with center x, but it may have measure zero (examples are easy to find, even among rectangles from $\mathcal{R}$, by looking at products that converge conditionally but not absolutely). So by taking the limit we just mean that the diameter of the rectangles goes to zero, where the distance is given by any metric compatible with the product topology, such as, for instance

$$d\big((x_n), (y_n)\big) = \sum_{n=0}^{\infty} \frac{(1/2^n)|x_n - y_n|}{1 + |x_n - y_n|}.$$

It makes no difference in the results that follow whether or not we require that the measure of the rectangles (in addition to their diameters) also go to zero. On the other hand, if we take the limit with respect to the collection of uncentered admissible rectangles containing a given point, then it is easy to see that in general the limit will not exist for integrable functions. Recall that $\mathcal{A}_0$ denotes the family of admissible rectangles centered at zero.

4.7. COROLLARY. *For every $f \in L^1(\mathbf{R}^\omega, \lambda^\omega)$ and almost every x, we have*

$$\lim_{\mathcal{A}_0} \frac{1}{\lambda^\omega(C)} \int_{x+C} f d\lambda^\omega = 0.$$

PROOF. Apply Theorem 4.5 to f^+ and f^-. $\square$

Thus, when differentiating integrals with respect to centered rectangles, the only function $f \in L^1(\mathbf{R}^\omega, \lambda^\omega)$ for which the limit gives the correct value is the constant zero. But the differentiation of integrals of continuous functions with respect to centered rectangles does recover the value of the function at each point.

4.8. THEOREM. *Let $f : \mathbf{R}^\omega \to \mathbf{R}$ be measurable. If there is a continuous function $g : \mathbf{R}^\omega \to \mathbf{R}$ with $f - g \in L^1(\mathbf{R}^\omega, \lambda^\omega)$, then for almost every $x \in \mathbf{R}^\omega$,*

$$\lim_{\mathcal{A}_0} \frac{1}{\lambda^\omega(C)} \int_{x+C} f d\lambda^\omega = g(x).$$

PROOF. Suppose first that f is continuous. Fix $x \in \mathbf{R}^\omega$ and $\varepsilon > 0$. Given any open neighborhood V of x for which $f(V) \subset \big(f(x) - \varepsilon, f(x) + \varepsilon\big)$, there exists a δ such that for every $C \in \mathcal{A}_0$, if the diameter of C is less than δ, then $x + C \subset V$. Thus

$$\frac{1}{\lambda^\omega(C)} \int_{x+C} \big| f - f(x) \big| d\lambda^\omega < \varepsilon.$$

The result for f measurable follows now from Corollary 4.7. $\square$

4.9. REMARK. We noted before that it was immaterial in the preceding results whether or not we required that the measure of the rectangles decreases to zero. On the other hand, this requirement does not, by itself, guarantee correct differentiation for continuous functions. Consider for instance the admissible rectangles of eccentricity bounded by a fixed $b > 1$, where the eccentricity of $\Pi_n I_n$ is defined as

$$\sup_{i,j} \frac{|I_i|}{|I_j|}.$$

Let $f\big((x_i)\big) = x_0^2$. Pick $x \in \mathbf{R}^\omega$, let C be any admissible rectangle centered at zero, of eccentricity bounded by b, and let a be the length of $\pi_0 C$. By Proposition 3.4, $\lim_n \pi_n C = 1$, so $a \geqq b^{-1}$. But now, by Fubini's Theorem

$$\frac{1}{\lambda^\omega(C)} \int_{x+C} f\, d\lambda^\omega = \frac{1}{a} \int_{x_0-a/2}^{x_0+a/2} y^2 dy = x_0^2 + \frac{a^2}{12} \geqq x_0^2 + \frac{1}{12b^2}.$$

Taking the lim inf as the measure of C decreases to zero (over admissible rectangles centered at zero, of eccentricity $\leqq b$), we see that the value obtained is too large for every $x \in \mathbf{R}^\omega$.

5. Final remarks

5.1. Remark. Let $\mu = \sum_1^n \delta_{x_i}$. If E is a measurable set, we denote by $\sharp E$ the number of point masses contained in E. The centered discrete Hardy–Littlewood maximal function associated to a family of rectangles $\mathcal{D}$ with center zero, is defined as

$$M_{\mathcal{D}}\mu(x) := \sup_{R \in \mathcal{D}} \frac{\sharp(x+R)}{\lambda^\omega(R)}.$$

Discretization results, which allow one to determine whether a maximal convolution operator satisfies a weak type (1,1) inequality by studying its action over finite sums of Dirac deltas, are well known in the $\mathbf{R}^n$ setting (see, for instance, [2], Theorem 4.1.1). However, such results do not extend to $\mathbf{R}^\omega$. We have seen that the Hardy–Littlewood maximal operator associated to centered rectangles maps every $L^1(\mathbf{R}^\omega, \lambda^\omega)$-function to the zero function. But it suffices one Dirac delta to verify that its discretized version is unbounded, even if we choose the family of rectangles $\mathcal{D}$ to be admissible rectangles from $\mathcal{R}$, of eccentricity $b > 1$, and with b as close to 1 as we wish. Using translates of the sets $U_n = \Pi_0^{n-1}(0,1) \times (0,b) \times \Pi_{n+1}^\infty(0,1)$, it is easy to check that $\lambda^\omega\big(\{M_{\mathcal{D}}\delta_0 \geqq b^{-1}\}\big) = \infty$.

5.2. Remark. Let ρ be the standard bounded metric on $\mathbf{R}$, i.e. $\rho(x,y) = \min\big\{|x-y|, 1\big\}$. It may be thought that the pathological results we have encountered are due to the fact that we are using the "wrong" topology on $\mathbf{R}^\omega$, so nonempty open sets have infinite measure and no nonzero function in $L^1(\mathbf{R}^\omega, \lambda^\omega)$ can be approximated by a continuous function. After all, if the relevant sets from a measure theoretic point of view are the rectangles, perhaps it is more natural to use the uniform topology, generated by the metric $u(x,y) = \sup_n \rho(x_n, y_n)$, and finer than the product topology. (Even finer, and still more natural in this context, is the box topology on $\mathbf{R}^\omega$, generated by the sets $\Pi_n(a_n, b_n)$.) But there cannot be a translation invariant Borel extension of λ^ω on $\mathbf{R}^\omega$ with the uniform topology. To see why, assume there exists such an extension, and call it ν. The set $A = \Pi_n(0, 1/2)$ is open in the uniform topology. Letting $B := \{ (x_n) : x_n = 0 \text{ or } x_n = 1/2 \}$, we have $\Pi_n[0,1) = \cup_{x \in B}(x + A) \cup \{ (x_n) \in \Pi_n[0,1)\colon \text{for some } n \in \omega,\ x_n = 0 \}$ $\cup \{ (x_n) \in \Pi_n[0,1)\colon \text{for some } n \in \omega,\ x_n = 1/2 \}$. It follows from Proposition 3.3 that the last two sets have measure zero, so $\nu\big(\cup_{x \in B} (x + A)\big) = 1$. Let $f : \cup_{x \in B}(x + A) \to 2^\omega$ be defined by $f(x + y) = 2x$ for every $y \in A$. Then the inverse image of each singleton is open and has measure zero, whence $\nu \circ f^{-1}$ is a continuous, translation invariant probability defined on all the subsets of the Cantor group 2^ω. But this contradicts Vitali's Theorem.

5.3. Remark. The space c_0 of real valued sequences with limit zero is contained in S_0^a for every $a > 0$ (in fact, it is clear that $c_0 = \cap_{n \geq 1} S_0^{1/n}$). So c_0 is λ^ω-measurable and $\lambda^\omega(c_0) = 0$. Hence, the same thing happens with all the spaces ℓ^p for $p \in (0, \infty)$. On the other hand, $\ell^\infty = \cup_n \big((-n, n)^\omega\big)$, so $\lambda^\omega(\ell^\infty) = \infty$.

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DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN
UNIVERSIDAD DE LA RIOJA
26004 LOGROÑO, LA RIOJA
SPAIN
E-MAIL: ALDAZ@DMC.UNIRIOJA.ES