ESTIMATES FOR COMMUTATORS OF ORTHOGONAL FOURIER SERIES

JOSÉ J. GUADALUPE, MARIO PÉREZ AND FRANCISCO J. RUIZ

ABSTRACT

In this paper we study weighted norm inequalities for the commutators $[b, S_n]$ where b is a BMO function and S_n denotes the nth partial sum of the Fourier series relative to a system of orthogonal polynomials on $[-1, 1]$ with respect to general weights. Results about generalized Jacobi and Bessel Fourier series are obtained.

0. *Introduction*

Given a linear operator T acting on functions and a function b, and denoting by M_b the operator of pointwise multiplication by $b(x)$, the commutator of this operator and T is defined by

$$[b, T]f(x) = [M_b, T]f(x) = b(x)\, Tf(x) - T(bf)(x).$$

The first results on this commutator were obtained by Coifman, Rochberg and Weiss [7]. They proved that if H is the classical Hilbert transform (and also for more general singular integrals) and $1 < p < \infty$, then $[b, H]$ is bounded in $L^p(\mathbb{R})$ if and only if $b \in \mathrm{BMO}(\mathbb{R})$. The boundedness of the commutator has been studied by, among others, Bloom [5] involving some weights and where b belongs to an appropriate weighted BMO space, and by Segovia and Torrea [24], who obtained a vector-valued commutator theorem for operators T including the Hilbert transform, and whose results apply to the Carleson operator, Littlewood–Paley sums, U.M.D. Banach spaces, parabolic differential equations and approximate identities (for further references see [25, 26]).

Frequently, the boundedness of the commutator is related to the analytic behaviour of some operator. Let $(X, d\mu)$ be a σ-finite measure space and B a real Banach function space on $(X, d\mu)$. Consider the partial sums of Fourier series relative to the orthonormal polynomials on $L^2(X, d\mu)$, that is, for $f \in L^2(X, d\mu)$ and $x \in X$, we have

$$S_n f(x) = \int_X K_n(x, y) f(y)\, d\mu(y),$$

where $K_n(x, y)$ is the corresponding kernel. Let $\bar{B}$ be the complexification of B and, for each $b \in \bar{B}$, define $T_n(b) = M_{e^b} S_n M_{e^{-b}}$. Then

$$T_n(b)(f)(x) = \int_X \exp[b(x) - b(y)]\, K_n(x, y) f(y)\, d\mu(y).$$

The boundedness of the operators $T_n(b)$ is equivalent to a weighted norm inequality for the operators S_n.

Received 30 July 1994.

1991 *Mathematics Subject Classification* 42C10.

Research supported by DGICYT and by IER.

J. London Math. Soc. (2) 54 (1996) 311–322

On the other hand, for this particular sequence $\{T_n\}_{n\in\mathbb{N}}$ of operator valued functions, the uniform boundedness in a neighbourhood of $0 \in \bar{B}$ implies the Gâteaux-differentiability (see [**6**, **15**]) and the Gâteaux-differential of T_n at 0 in the direction $b \in \bar{B}$ is

$$\frac{d}{dz} T_n(zb)\,|_{z=0} = [b, S_n].$$

In particular, these ideas (which came out in [**7**]) show that certain weighted norm inequalities for a basic operator T give information about the commutator $[b, T]$.

The purpose of this paper is to study the uniform boundedness of the commutator of the partial sums of Fourier series with respect to a class of weights which includes, as a particular case, generalized Jacobi Fourier series.

In the first step, we find necessary conditions for the uniform boundedness of the operators $[b, S_n]$ in $L^p(d\mu)$; also, for the uniform weak boundedness $[b, S_n]$: $L^p(d\mu) \to L^{p,\infty}(d\mu)$ or the restricted weak boundedness $[b, S_n]$: $L^{p,1}(d\mu) \to L^{p,\infty}(d\mu)$.

Here, $L^{p,r}(d\mu)$ stands for the classical Lorentz space of all measurable functions f satisfying

$$\|f\|_{L^{p,r}(d\mu)} = \left(\frac{r}{p}\int_0^\infty [t^{1/p} f^*(t)]^r \frac{dt}{t}\right)^{1/r} < \infty \quad \text{for } 1 \leqslant p < \infty, 1 \leqslant r < \infty,$$

$$\|f\|_{L^{p,\infty}(d\mu)} = \|f\|_{L^p_*(d\mu)} = \sup_{t>0} t^{1/p} f^*(t) < \infty \quad \text{for } 1 \leqslant p \leqslant \infty,$$

where f^* denotes the nonincreasing rearrangement of f. We refer the reader to [**27**] for further information on these topics.

In the second step, we find sufficient conditions for the uniform boundedness of $[b, S_n]$ in $L^p(d\mu)$, which, in many cases, coincide with the necessary conditions previously found. We are concerned with the case $d\mu = w dx$, where w is a positive weight function.

We shall distinguish two cases: firstly, polynomial systems with uniform bounds (the class $\mathscr{H}$ defined below), where we follow the ideas of Coifman, Rochberg and Weiss. This is the case of Jacobi weights $(1-x)^\alpha (1+x)^\beta$ with $\alpha, \beta \geqslant -\frac{1}{2}$. Fourier Bessel series also fall in this scheme.

Next, we consider a more general setting (the class $\bar{\mathscr{H}}$), where these techniques do not work well and a more detailed examination of the kernels is required. Here, we reduce the problem to the boundedness of $[H, b]$ (where H is the Hilbert transform) in weighted L^p spaces, by inserting A_p weights.

We shall need weighted estimates for the partial sums of the Fourier series. The problem of finding conditions on weights u, v such that

$$\|u S_n f\|_{L^p(w)} \leqslant C \|v f\|_{L^p(w)} \quad \forall n \geqslant 0, \forall f \in L^p(v^p w) \tag{1}$$

has been solved only in some particular cases. For instance, Badkov gives in [**2**] necessary and sufficient conditions for (1) when $u = v$ and both u, w are generalized Jacobi weights; earlier results can be found in [**21**, **22**, **28**, **18**]. For the two weight case, see [**12**, **9**, **10**]. Hermite and Laguerre series have been considered by Askey and Wainger [**1**] and Muckenhoupt [**19**].

Bessel series have been studied by Wing [**28**], Benedek-Panzone [**3**, **4**] and the authors [**11**].

In this paper we shall see that the boundedness of the commutator for this type of series holds when the partial sums of the Fourier series are bounded.

1. *Notation and main results*

Let $d\mu(x) = w(x)\,dx$, with $w \in L^1(dx)$ and $w > 0$ a.e. in $[-1,1]$. Let $\{p_n\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to μ.

For $f \in L^1(w)$, let $S_n f$ denote the nth partial sum of the Fourier expansion of f in $\{p_n\}_{n\geq 0}$, that is,

$$S_n f(x) = \int_{\mathbb{R}} f(y)\,K_n(x,y)\,w(y)\,dy, \quad K_n(x,y) = \sum_{k=0}^{n} p_k(x)\,p_k(y).$$

Throughout this paper, C will denote a constant, independent of n, f, but possibly different from line to line.

Let $1 < p < \infty$ and $-\infty \leqslant a < b \leqslant \infty$. The class $A_p(a,b)$ consists of those pairs of weights (u,v) such that

$$\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I v(x)^{-1/(p-1)}dx\right)^{p-1} \leqslant C,$$

where I ranges over all finite intervals $I \subseteq (a,b)$ and $|I|$ stands for the length of the interval I. A weight u is said to belong to A_p if $(u,u) \in A_p$. We refer the reader to **[8]** for further details on A_p classes.

We say that $(u,v) \in A_p^\delta(a,b)$ for $\delta > 1$ if $(u^\delta, v^\delta) \in A_p(a,b)$. With this definition we mean that a power of u and v greater than 1 belongs to A_p. We use the same exponent δ although it can change in each ocurrence.

We shall take B to be the space of functions of bounded mean oscillation (BMO) on $[-1,1]$. If $b \in L^1(dx)$, the mean of b on an interval I is

$$b_I = \frac{1}{|I|}\int_I b(x)\,dx.$$

The function b is said to have bounded mean oscillation on $[-1,1]$ if

$$\|b\|_* = \sup_I \frac{1}{|I|}\int_I |b(\theta) - b_I|\,d\theta$$

is finite, where the supremum is taken over all intervals $I \subseteq [-1,1]$. The space BMO of real-valued functions (modulo constants) having bounded mean oscillation on $[-1,1]$ is a Banach space with $\|\cdot\|_*$ as its norm.

THEOREM 1. *Let w be a weight on $[-1,1]$ with $w > 0$ a.e., $\{S_n\}_{n\geq 0}$ the Fourier series relative to $d\mu(x) = w(x)\,dx$, U, V two weights, $U, V^{-1} > 0$ a.e. Let $b \in$ BMO, $b \notin L^\infty$ and suppose that there exists some constant $C > 0$ with*

$$\|U[b,S_n](V^{-1}f)\|_{L^{p,r}(w)} \leqslant C\|f\|_{L^{q,s}(w)}$$

for each $n \geqslant 0$ and $f \in L^{q,s}(w)$ (where $1 < p < \infty$, $1 < q < \infty$; either $r = p$ or $r = \infty$; either $s = q$ or $s = 1$). Then,

$$\|bUw^{-1/2}(1-x^2)^{-1/4}\|_{L^{p,r}(w)} < \infty,$$
$$\|bV^{-1}w^{-1/2}(1-x^2)^{-1/4}\|_{L^{q',s'}(w)} < \infty.$$

Let $w(x)$ be a weight function on $[-1,1]$, $p_n(x)$ the corresponding orthonormal polynomials and $q_n(x)$ the orthonormal polynomials with respect to $(1-x^2)\,w(x)$. We say that w belongs to the class $\mathscr{H}$ of weights if it satisfies

(i) $w(x) > 0$ a.e.,
(ii) $|p_n(x)| \leqslant Cw(x)^{-1/2}(1-x^2)^{-1/4}$,
(iii) $|q_n(x)| \leqslant Cw(x)^{-1/2}(1-x^2)^{-3/4}$.
The class $\mathscr{H}$ contains the generalized Jacobi weights

$$w(x) = \phi(x)(1-x)^{\alpha}(1+x)^{\beta}\prod_{i=1}^{N}|x-x_i|^{\gamma_i},$$

where $\alpha, \beta \geqslant -\frac{1}{2}$, $\gamma_i \geqslant 0$ (for $i = 1, 2, \ldots, N$), $-1 < x_1 < \cdots < x_n < 1$, ϕ is positive and continuous on $[-1, 1]$ and $\rho(\delta)/\delta \in L^1(0, 2)$, ρ being the modulus of continuity of ϕ (see, for example, [**2**]).

THEOREM 2. *Let* $1 < p < \infty$, $w \in \mathscr{H}$, *U and V be weights on* $[-1, 1]$ *and* $b \in \mathrm{BMO}$. *If*

$$((1-x^2)^{-p/4}U^p w^{1-p/2}, (1-x^2)^{-p/4}V^p w^{1-p/2}) \in A_p^{\delta}(-1, 1),$$
$$((1-x^2)^{p/4}U^p w^{1-p/2}, (1-x^2)^{p/4}V^p w^{1-p/2}) \in A_p^{\delta}(-1, 1) \tag{2}$$

for some $\delta > 1$ (*where* $\delta = 1$ *when* $U = V$), *then the commutator* $[b, S_n]$ *is bounded from* $L^p(V^p w)$ *into* $L^p(U^p w)$ *uniformly in n.*

For generalized Jacobi weights with $\alpha, \beta > -1$, $\gamma_i \geqslant 0$, the orthogonal polynomials do not have uniform bounds. We extend the class $\mathscr{H}$ of weights and say that a weight w belongs to the class $\bar{\mathscr{H}}$ if $w(x) = (1-x)^{\alpha}(1+x)^{\beta}w_1(x)$, where

(i) $w(x) > 0$ a.e. and there exist $\varepsilon > 0$ and positive constants C_1 and C_2 such that $C_1 < w_1(x) < C_2$ for all $x \in (1-\varepsilon, 1)$ and $x \in (-1, -1+\varepsilon)$,
(ii) $|p_n(x)| \leqslant C(1-x+a_n)^{-(\alpha/2+1/4)}(1+x+b_n)^{-(\beta/2+1/4)}w_1(x)^{-1/2}$,
(iii) $|q_n(x)| \leqslant C(1-x+a_n)^{-(\alpha/2+3/4)}(1+x+b_n)^{-(\beta/2+3/4)}w_1(x)^{-1/2}$, where $\{a_n\}$ and $\{b_n\}$ are positive sequences such that $\lim_n a_n = \lim_n b_n = 0$.

THEOREM 3. *Let* $1 < p < \infty$, $w \in \bar{\mathscr{H}}$, $U(x) = (1-x)^a(1+x)^b u(x)$, $V(x) = (1-x)^A (1+x)^B v(x)$ *with* $u > 0$ a.e., $v > 0$ a.e. *and such that* $C_1 < u(x)$, $v(x) < C_2$ *for* $x \in (1-\varepsilon, 1)$ *and* $x \in (-1, -1+\varepsilon)$. *If* $b \in \mathrm{BMO}$,

$$\left|(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{a+A}{2}\right| < \frac{a-A}{2}+\min\left\{\frac{1}{4}, \frac{\alpha+1}{2}\right\} \quad \textit{for } A \leqslant a,$$

$$\left|(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{b+B}{2}\right| < \frac{b-B}{2}+\min\left\{\frac{1}{4}, \frac{\beta+1}{2}\right\} \quad \textit{for } B \leqslant b,$$

and

$$(u^p w_1^{1-p/2}, v^p w_1^{1-p/2}) \in A_p^{\delta}(-1, 1)$$

for some $\delta > 1$ (*with* $\delta = 1$ *when* $u = v$), *then the commutator* $[b, S_n]$ *is bounded from* $L^p(V^p w)$ *into* $L^p(U^p w)$ *uniformly in n.*

As a consequence of these results for generalized Jacobi weights, we obtain the following.

COROLLARY 1. *Let* $1 < p < \infty$, $w(x) = (1-x)^\alpha(1+x)^\beta \prod_{i=1}^N |x-x_i|^{\gamma_i}$ *with* $x_i \in (-1,1)$, $x_i \neq x_j \forall i \neq j$, $\alpha, \beta > -1$, $\gamma_i \geqslant 0 \forall i$ *and*

$$U(x) = (1-x)^a(1+x)^b \prod_{i=1}^N |x-x_i|^{g_i}.$$

Then the commutator $[b, S_n]$ is uniformly bounded from $L^p(U^p w)$ into $L^p(U^p w)$ for each $b \in BMO$ *if and only if*

$$\left| a+(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right) \right| < \min\left\{\frac{1}{4}, \frac{\alpha+1}{2}\right\}, \tag{3}$$

$$\left| b+(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right) \right| < \min\left\{\frac{1}{4}, \frac{\beta+1}{2}\right\} \tag{4}$$

and

$$\left| g_i+(\gamma_i+1)\left(\frac{1}{p}-\frac{1}{2}\right) \right| < \min\left\{\frac{1}{2}, \frac{\gamma_i+1}{2}\right\} \quad \text{for } i = 1, 2, \ldots, N. \tag{5}$$

COROLLARY 2. *With the same notation, the inequalities* (3), (4), (5) *are also necessary for the weak and restricted weak* (p,p)*-boundedness of the commutator* $[b, S_n]$ *for each* $b \in$ BMO.

REMARK. Notice that, in contrast to this situation, the operators S_n are of restricted weak type when w is a Jacobi weight and p is an endpoint of the open interval determined by (3), (4), (5) (see [13]).

2. *Proofs of the theorems*

Proof of Theorem 1. For each $0 < L < K < \infty$, let us define

$$\mathcal{P}(L) = \{x \in [-1,1]; |b(x)| < L\}, \quad \mathcal{G}(K) = \{x \in [-1,1]; K < |b(x)|\}.$$

Then, for each $x \in \mathcal{G}(K)$, $y \in \mathcal{P}(L)$ we have

(*) $\operatorname{sgn}(b(x)-b(y)) = \operatorname{sgn} b(x)$;

(*) $|b(y)| < L < (L/K)|b(x)|$, so that

$$|b(x)-b(y)| \geqslant |b(x)|-|b(y)| > ((K-L)/K)\,|b(x)|.$$

From the hypothesis it follows that

$$\|u[b, S_n - S_{n-1}](v^{-1}f)\|_{L^{p,r}(w)} \leqslant C\|f\|_{L^{q,s}(w)},$$

$$[b, S_n - S_{n-1}](v^{-1}f)(x) = p_n(x)\int_{-1}^{1}[b(x)-b(y)]p_n(y)v(y)^{-1}f(y)w(y)\,dy,$$

where the $\{p_n\}$ are the orthonormal polynomials with respect to $w(x)\,dx$. Now, take $0 < L < K < \infty$ and $f(y) = [\operatorname{sgn} p_n(y)]\chi_{\mathcal{P}(L)}(y)|h(y)|$, where h is any function in $L^{q,s}(w)$. Here and in the sequel, χ_A denotes the characteristic function on a measurable set A. For each $x \in \mathcal{G}(K)$, we have

$$\begin{aligned} &|[b, S_n - S_{n-1}](v^{-1}f)(x)| \\ &= |-p_n(x)\operatorname{sgn} b(x)| \int_{-1}^{1} |b(y)-b(x)|\,|p_n(y)|\,v(y)^{-1}\chi_{\mathcal{P}(L)}(y)\,|h(y)|\,w(y)\,dy \\ &\geqslant \frac{K-L}{K}|p_n(x)|\,|b(x)|\,\|p_n v^{-1}\chi_{\mathcal{P}(L)} h\|_{L^1(w)}. \end{aligned}$$

Thus,

$$\|u[b, S_n - S_{n-1}](v^{-1}f)\|_{L^{p,r}(w)} \geq \frac{K-L}{K} \|\chi_{\mathscr{G}(K)} bup_n\|_{L^{p,r}(w)} \|p_n v^{-1} \chi_{\mathscr{P}(L)} h\|_{L^1(w)}$$

and therefore

$$\frac{K-L}{K} \|\chi_{\mathscr{G}(K)} bup_n\|_{L^{p,r}(w)} \|p_n v^{-1} \chi_{\mathscr{P}(L)} h\|_{L^1(w)} \leq C\|f\|_{L^{q,s}(w)} \leq C\|h\|_{L^{q,s}(w)}$$

for each $h \in L^{q,s}(w)$. By duality,

$$\frac{K-L}{K} \|\chi_{\mathscr{G}(K)} bup_n\|_{L^{p,r}(w)} \|p_n v^{-1} \chi_{\mathscr{P}(L)}\|_{L^{q',s'}(w)} \leq C.$$

Also,

$$\frac{K-L}{KL} \|\chi_{\mathscr{G}(K)} bup_n\|_{L^{p,r}(w)} \|\chi_{\mathscr{P}(L)} bv^{-1} p_n\|_{L^{q',s'}(w)} \leq C. \tag{6}$$

In a similar way, taking $f(y) = [\operatorname{sgn} b(y)] \operatorname{sgn} p_n(y) \chi_{\mathscr{G}(K)}(y) |h(y)|$, and $x \in \mathscr{P}(L)$, we obtain

$$\frac{K-L}{KL} \|\chi_{\mathscr{P}(L)} bup_n\|_{L^{p,r}(w)} \|\chi_{\mathscr{G}(K)} bv^{-1} p_n\|_{L^{q',s'}(w)} \leq C. \tag{7}$$

Now, by a result of Máté, Nevai and Totik (see [17]),

$$C\|gw^{-1/2}(1-x^2)^{-1/4}\|_{L^p(w)} \leq \liminf_n \|gp_n\|_{L^p(w)}$$

for any measurable function g. A similar property holds in $L^{p,\infty}(w)$ (see [13]). Then, taking lim inf in (6) and (7) we have

$$\|\chi_{\mathscr{G}(K)} buw^{-1/2}(1-x^2)^{-1/4}\|_{L^{p,r}(w)} \|\chi_{\mathscr{P}(L)} bv^{-1}w^{-1/2}(1-x^2)^{-1/4}\|_{L^{q',s'}(w)} < \infty,$$
$$\|\chi_{\mathscr{P}(L)} buw^{-1/2}(1-x^2)^{-1/4}\|_{L^{p,r}(w)} \|\chi_{\mathscr{G}(K)} bv^{-1}w^{-1/2}(1-x^2)^{-1/4}\|_{L^{q',s'}(w)} < \infty$$

for each $0 < L < K < \infty$. Since $b \notin L^\infty$ we have $\chi_{\mathscr{G}(K)} b \neq 0$ for every $K > 0$ and there exists some $L_0 > 0$ such that $\chi_{\mathscr{P}(L)} b \neq 0$ for every $L > L_0$. Now, $u, v^{-1} > 0$ almost everywhere, so that for $L_0 < L < K < \infty$ the above norms cannot vanish and as a consequence they cannot be ∞ either. This proves the theorem.

Proof of Theorem 2. Write

$$u(x) = (1-x^2)^{-p/4} U(x)^p w(x)^{1-p/2}, \quad v(x) = (1-x^2)^{-p/4} V(x)^p w(x)^{1-p/2},$$
$$\bar{u}(x) = (1-x^2)^{p/4} U(x)^p w(x)^{1-p/2}, \quad \bar{v}(x) = (1-x^2)^{p/4} V(x)^p w(x)^{1-p/2}.$$

The following lemmas will be proved below.

LEMMA 1. *Assume that $w \in \mathscr{H}$ and let U and V be as above and satisfying $(u, v) \in A_p^\delta$ and $(\bar{u}, \bar{v}) \in A_p^\delta$ for some $\delta > 1$. Then $\|US_n f\|_{p,w} \leq C\|Vf\|_{p,w}$, where C depends only on the A_p constants of (u, v) and $(\bar{u}, \bar{v})$.*

LEMMA 2. *Let $(u_1, v_1) \in A_p^\delta$ for some $\delta > 1$ and $b \in \mathrm{BMO}$. Then there exist $\delta > 1$ and $\gamma > 0$ such that $(e^{sb}u_1, e^{sb}v_1) \in A_p^\delta$ for all s with $|s| < \gamma$, and the A_p constant is independent of s.*

Now, for a fixed function $b \in \mathrm{BMO}$ and $n \in \mathbb{N}$, put $T_z f = e^{zb} S_n(e^{-zb} f)$ for $z \in \mathbb{C}$. Let us show the analyticity of this operator-valued function. From the hypothesis and Lemma 2 it follows that $(e^{sb}u, e^{sb}v) \in A_p^\delta$ and $(e^{sb}\bar{u}, e^{sb}\bar{v}) \in A_p^\delta$ for all s such that $|s| < \gamma$. Then, by Lemma 1, we have $\|e^{sb} U S_n f\|_{p,w} \leqslant C \|e^{sb} V f\|_{p,w}$. Therefore, for $|z| < \gamma$ we have $\|U T_z f\|_{p,w} \leqslant C \|V f\|_{p,w}$. Then, $T_z \in \mathscr{L}(L^p(V^p w), L^p(U^p w))$ for $|z| < \gamma$. Moreover, the constant C in the last inequality is independent of z for $|z| < \gamma$. So, the application T_z is bounded (with the operator norm) in $|z| < \gamma$. Then, in order to prove the analyticity in $|z| < \gamma$ it is enough to show that the mapping $z \mapsto \langle T_z f, g \rangle$ is holomorphic for every f in a dense subspace of $L^p(V^p w)$ and every g in a dense subspace of the dual of $L^p(U^p w)$ (see [14, p. 365]).

If f, g are bounded functions we can differentiate the expression

$$\langle T_z f, g \rangle = \int_{-1}^{1} \int_{-1}^{1} e^{z(b(x)-b(y))} K_n(x,y) f(x) g(y)\, U(x)^p\, w(x)\, w(y)\, dx\, dy$$

by differentiating under the integral sign, since the derivative of the integrand can be dominated by

$$C e^{\gamma |b(x)-b(y)|} |b(x)-b(y)|\, |K_n(x,y)|\, U(x)^p\, w(x)\, w(y),$$

which is integrable on $[-1,1] \times [-1,1]$. This follows from a suitable handling of the hypothesis (integrability conditions which are implicit in the A_p^δ conditions (2), $b \in \mathrm{BMO}$ and $w \in \mathscr{H}$).

Besides, this process shows that

$$\frac{d}{dz} T_z |_{z=0} = [b, S_n].$$

Therefore, $[b, S_n]$ is a bounded operator from $L^p(V^p w)$ into $L^p(U^p w)$. Moreover, by Cauchy's integral theory, the norm of $[b, S_n]$ is controlled by the maximum of the norms of T_z (which are independent of n), when z ranges in a circle, and hence the norms of $[b, S_n]$ are independent of n. This concludes the proof of Theorem 2.

Proof of Lemma 1. The main idea of this proof comes from [21] (see also [12]). We use Pollard's decomposition of the kernels $K_n(x,y)$, that is,

$$K_n(x,y) = r_n\, T_{1,n}(x,y) + s_n\, T_{2,n}(x,y) + s_n\, T_{3,n}(x,y),$$

where

$$T_{1,n}(x,y) = p_{n+1}(x)\, p_{n+1}(y),$$

$$T_{2,n}(x,y) = (1-y^2) \frac{p_{n+1}(x)\, q_n(y)}{x-y}, \quad T_{3,n}(x,y) = (1-x^2) \frac{p_{n+1}(y)\, q_n(x)}{y-x}$$

and $\{r_n\}$, $\{s_n\}$ are bounded sequences. In fact, for any measure $d\mu$ on $[-1,1]$ with $\mu' > 0$ a.e.,

$$\lim_n r_n = -\tfrac{1}{2}, \quad \lim_n s_n = \tfrac{1}{2}$$

(this can be deduced from [21] and either [23] or [16]). Therefore, we can write $S_n f = r_n W_{1,n} f + s_n W_{2,n} f - s_n W_{3,n} f$, where

$$W_{1,n} f(x) = p_{n+1}(x) \int_{-1}^{1} p_{n+1} f w,$$

$$W_{2,n} f(x) = p_{n+1}(x)\, H((1-y^2)\, q_n f w, x), \quad W_{3,n} f(x) = (1-x^2)\, q_n(x)\, H(p_{n+1} f w, x),$$

and H is the Hilbert transform on the interval $[-1,1]$. Thus, the study of S_n can be reduced to that of $W_{i,n}$ (for $i = 1, 2, 3$).

Case $i = 1$. By using the uniform estimates for p_n and q_n and Hölder's inequality with $1/p + 1/p' = 1$, we have

$$\|UW_{1,n}f\|_{p,w} = \|Up_{n+1}\|_{p,w} \left| \int_{-1}^{1} p_{n+1}(y) f(y)\, w(y)\, dy \right|$$
$$\leqslant C\|U(x)\, w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p,w}\, \|V(x)^{-1}\, w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p',w}\, \|Vf\|_{p,w}.$$

From the A_p conditions in the hypothesis it follows that

$$U(x)^p\, w(x)^{1-p/2}(1-x^2)^{-p/4} \in L^1(dx), \quad (V(x)^p\, w(x)^{1-p/2}(1-x^2)^{p/4})^{-p'/p} \in L^1(dx),$$

that is,

$$\|U(x)\, w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p,w} < \infty, \quad \|V(x)^{-1}\, w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p',w} < \infty.$$

Therefore $\|UW_{1,n}f\|_{p,w} \leqslant C\|Vf\|_{p,w}$.

Case $i = 2$. Since $((1-x^2)^{-p/4} w^{1-p/2} U^p, (1-x^2)^{-p/4} w^{1-p/2} V^p) \in A_p^{\delta}(-1, 1)$ for some $\delta > 1$, the Hilbert transform is bounded from $L^p((1-x^2)^{-p/4} w^{1-p/2} V^p)$ into $L^p((1-x^2)^{-p/4} w^{1-p/2} U^p)$ (it is a consequence of [**20**, Theorem 3]).

Write $g(y) = (1-y^2)\, q_n(y) f(y)\, w(y)$. Then

$$\|UW_{2,n}f\|_{p,w} = \|Up_{n+1} Hg\|_{p,w} \leqslant C\|U(1-x^2)^{-1/4}\, w^{-1/2} Hg\|_{p,w}$$
$$\leqslant C\|V(1-x^2)^{-1/4}\, w^{-1/2} g\|_{p,w} \leqslant C\|Vf\|_{p,w}.$$

Case $i = 3$. This can be done in a similar way using the second A_p^{δ}-condition.

Proof of Lemma 2. For any interval I, write $I(f) = (1/|I|)\int_I f(x)\, dx$. The condition $(u_1, v_1) \in A_p^{\delta}$ can be written as $I(u_1^{\delta})\, I(v_1^{-\delta/(p-1)})^{p-1} \leqslant C$ for each interval. It is known [**20**] that there exists $\delta > 1$ such that $(u_1^{\delta}, v_1^{\delta}) \in A_p$ if and only if there is some $\sigma \in A_p$ with $C_1 u_1 \leqslant \sigma \leqslant C_2 v_1$, where δ and the A_p constants depend on each other. In order to prove that $(e^{sb}u_1, e^{sb}v_1) \in A_p^{\delta}$ it is enough to show that $e^{sb}\sigma \in A_p$ uniformly in s.

Since $\sigma \in A_p$, by the reverse Hölder's inequality there exists $\varepsilon > 1$ such that $\sigma^{\varepsilon} \in A_p$. As $b \in \mathrm{BMO}$ (and also $-b \in \mathrm{BMO}$), by the John–Nirenberg inequality there exists $\lambda > 0$ small enough such that $e^{sb} \in A_p$ for $|s| < \lambda$ uniformly in s, that is, with an A_p constant independent of s (see [**8**]).

By Hölder's inequality with $1/\varepsilon + 1/\varepsilon' = 1$, we have

$$I(e^{sb}\sigma) \leqslant I(\sigma^{\varepsilon})^{1/\varepsilon}\, I(e^{\varepsilon' sb})^{1/\varepsilon'}, \quad I((e^{sb}\sigma)^{-1/(p-1)}) \leqslant I(\sigma^{-\varepsilon/(p-1)})^{1/\varepsilon}\, I(e^{-\varepsilon' sb/(p-1)})^{1/\varepsilon'}.$$

Therefore

$$I(e^{sb}\sigma)\, I((e^{sb}\sigma)^{-1/(p-1)})^{p-1} \leqslant [I(\sigma^{\varepsilon})\, I(\sigma^{-\varepsilon/(p-1)})]^{1/\varepsilon}\, [I(e^{\varepsilon' sb})\, I(e^{-\varepsilon' sb/(p-1)})]^{1/\varepsilon'} \leqslant C$$

for every s such that $|s| < \lambda/\varepsilon'$.

LEMMA 3. *Let* $R, S \in \mathbb{R}$, $a_n > 0$, $\lim_n a_n = 0$, $t \in [-1, 1]$. *Then,*

(a) $|x-t|^R(|x-t| + a_n)^S \in A_p(-1, 1)$ *uniformly in n if and only if* $-1 < R < p-1$, $-1 < R+S < p-1$;

(b) *for a product of terms of this type, these conditions are applied separately to each factor.*

For the proof of this lemma, see [**13**].

Proof of Theorem 3. Coming back again to Pollard's decomposition we have $[b, S_n] = \sum_{i=1}^{3} [b, W_{i,n}]$, where

$$[b, W_{1,n}]f = bp_{n+1} \int_{-1}^{1} p_{n+1} fw - p_{n+1} \int_{-1}^{1} p_{n+1} fbw,$$

$$[b, W_{2,n}]f = p_{n+1}[b, H]((1-y^2) q_n fw), \quad [b, W_{3,n}]f = (1-x^2) q_n [b, H](p_{n+1} fw).$$

We consider each operator separately.

(i) *Boundedness of* $[b, W_{2,n}]$. Write

$$\lambda_n = U^p |p_{n+1}|^p w \quad \text{and} \quad \mu_n = V^p |q_n|^{-p} (1-x^2)^{-p} w^{1-p}.$$

Now $\| U[b, W_{2,n}]f \|_{p,w} \leqslant C \| Vf \|_{p,w}$ if and only if $\| [b, H] g \|_{p,\lambda_n} \leqslant C \| g \|_{p,\mu_n}$ with some constant C independent of n. In order to prove this last inequality, we use the idea of inserting weights ϕ_n, that is, finding functions ϕ_n such that $C_1 \lambda_n \leqslant \phi_n \leqslant C_2 \mu_n$ and $\phi_n \in A_p$ uniformly, that is, with an A_p-constant independent of n. By using the estimates for p_n and q_n we have

$$\lambda_n \leqslant C u^p w_1^{1-p/2} (1-x)^{ap+\alpha} (1+x)^{bp+\beta} (1-x+a_n)^{-p(\alpha/2+1/4)} (1+x+b_n)^{-p(\beta/2+1/4)},$$

$$\mu_n \geqslant C v^p w_1^{1-p/2} (1-x)^{Ap-\alpha+\alpha(1-p)} (1+x)^{Bp-\beta+\beta(1-p)} \times (1-x+a_n)^{p(\alpha/2+3/4)} (1+x+b_n)^{p(\beta/2+3/4)}.$$

It is not difficult to see, from the hypothesis, that we can take a real number R such that $Ap - p + \alpha(1-p) \leqslant R \leqslant ap + \alpha$ with $-1 < R < p-1$ and choose S such that

$$Ap - p + \alpha(1-p) + p(\tfrac{1}{2}\alpha + \tfrac{3}{4}) \leqslant R + S \leqslant ap + \alpha - p(\tfrac{1}{2}\alpha + \tfrac{1}{4}),$$
$$-1 < R + S < p - 1.$$

Now, it is a straightforward calculation to verify that

$$C(1-x)^{ap+\alpha} (1-x+a_n)^{-p(\alpha/2+1/4)} \leqslant (1-x)^R (1-x+a_n)^S \leqslant C(1-x)^{Ap-p+\alpha(1-p)} (1-x+a_n)^{p(\alpha/2+3/4)}.$$

We can also take $\tilde{R}$ and $\tilde{S}$ such that

$$Bp - p + \beta(1-p) \leqslant \tilde{R} \leqslant bp + \beta,$$
$$Bp - p + \beta(1-p) + p(\tfrac{1}{2}\beta + \tfrac{3}{4}) \leqslant \tilde{R} + \tilde{S} \leqslant bp + \beta - p(\tfrac{1}{2}\beta + \tfrac{1}{4}),$$
$$-1 < \tilde{R} < p-1, \quad -1 < \tilde{R} + \tilde{S} < p - 1,$$

so that

$$C(1+x)^{bp+\beta} (1+x+b_n)^{-p(\beta/2+1/4)} \leqslant (1+x)^{\tilde{R}} (1+x+b_n)^{\tilde{S}} \leqslant C(1+x)^{Bp-p+\beta(1-p)} (1+x+b_n)^{p(\beta/2+3/4)}.$$

If we write $\alpha_n(x) = (1-x)^R (1-x+a_n)^S$ and $\beta_n(x) = (1+x)^{\tilde{R}} (1+x+b_n)^{\tilde{S}}$ we have $C\lambda_n \leqslant u^p w_1^{1-p/2} \alpha_n \beta_n$ and $v^p w_1^{1-p/2} \alpha_n \beta_n \leqslant C \mu_n$. As $(u^p w_1^{1-p/2}, v^p w_1^{1-p/2}) \in A_p^\delta$, there exists a positive function ϕ satisfying $C_1 u^p w_1^{1-p/2} \leqslant \phi \leqslant C_2 v^p w_1^{1-p/2}$ and $\phi^p w_1^{1-p/2} \in A_p$. Besides, there are positive constants C_1 and C_2 such that $C_1 \leqslant \phi(x) \leqslant C_2$ for all $x \in (-1, -1+\varepsilon)$ and $x \in (1-\varepsilon, 1)$. On the other hand, having in mind that $a_n > 0$, $\lim_n a_n = 0$ and $-1 < R < p-1$, $-1 < R+S < p-1$, from Lemma 3 it follows that

$$\alpha_n = (1-x)^R (1-x+a_n)^S \in A_p \quad \text{uniformly.}$$

Also, it is clear that α_n is bounded below and above by positive constants on the interval $[-1, 1-\varepsilon]$. In a similar way we obtain that $\beta_n \in A_p$ uniformly and there exist positive constants C_1 and C_2 such that $C_1 < \beta(x) < C_2$ for all $x \in [-1+\varepsilon, 1]$. Then, splitting in pieces the integrals appearing in the A_p condition it can be shown that $\phi_n = \phi^p w_1^{1-p/2} \alpha_n \beta_n \in A_p$ uniformly. Since the commutator of the Hilbert transform with a function $b \in \mathrm{BMO}$ is bounded with A_p weights (see [5]), then

$$\|[b, H]g\|_{p,\lambda_n} \leqslant C\|[b, H]g\|_{p,\phi_n} \leqslant C_1\|g\|_{p,\phi_n} \leqslant C_2\|g\|_{p,\mu_n}$$

and the boundedness of $[b, W_{2,n}]$ follows.

(ii) *Boundedness of* $[b, W_{3,n}]$. We can prove that there are positive constants C_1, C_2 and weights ψ_n uniformly in A_p, such that

$$C_1\, U(x)^p (1-x^2)^p |q_n(x)|^p w(x) \leqslant \psi_n(x) \leqslant C_2\, V(x)^p |p_{n+1}(x)|^{-p} w(x)^{1-p},$$
$$\psi_n \in A_p \quad \text{uniformly},$$

and we proceed as before.

(iii) *Boundedness of* $[b, W_{1,n}]$. We have $[b, W_{1,n}]f = A_n f + B_n f$, where

$$A_n f = (b - b_Q) p_{n+1} \int_{-1}^{1} p_{n+1} f w, \quad B_n f = p_{n+1} \int_{-1}^{1} (b - b_Q) p_{n+1} f w$$

and Q stands for the interval $[-1, 1]$. Moreover,

$$\|UA_n f\|_{p,w} = \|(b-b_Q)\, Up_{n+1}\|_{p,w} \left| \int_{-1}^{1} p_{n+1} f w \right|$$
$$\leqslant \|(b-b_Q)\, Up_{n+1}\|_{p,w} \|p_{n+1} V^{-1}\|_{p',w} \|Vf\|_{p,w}.$$

Let $\delta > 1$ satisfying the A_p hypothesis, $\varepsilon > 0$, and $1/p = 1/s + 1/p\delta + 1/p(1+\varepsilon)$. From the definitions of λ_n, α_n, β_n and Hölder's inequality we have

$$\|(b-b_Q)\, Up_{n+1}\|_{p,w} = \|(b-b_Q)\, \lambda_n^{1/p}\|_p \leqslant \|(b-b_Q)\, [u^p w_1^{1-p/2}]^{1/p} \alpha_n^{1/p} \beta_n^{1/p}\|_p$$
$$\leqslant \|(b-b_Q)\|_s \|[u^p w_1^{1-p/2}]^\delta\|_1^{1/(p\delta)} \|\alpha_n \beta_n\|_{1+\varepsilon}^{1/p}.$$

From the A_p hypothesis, $\|[u^p w_1^{1-p/2}]^\delta\|_1^{1/(p\delta)} < C$. Now, $\varepsilon > 0$ can be taken small enough so that $\|\alpha_n \beta_n\|_{1+\varepsilon}^{1/p} < C$. Finally, from the John–Nirenberg theorem, there exists some C such that $\|(b-b_Q)\|_s \leqslant C\|b\|_*$. Putting these inequalities together, it follows that $\|(b-b_Q)\, Up_{n+1}\|_{p,w} < C$. In an analogous way $\|p_{n+1} V^{-1}\|_{p',w} < C$. Thus

$$\|UA_n f\|_{p,w} \leqslant C\|Vf\|_{p,w}.$$

The operators $B_n f$ can be handled in the same way as before.

Proof of Corollaries 1 *and* 2. (a) If $r \in \mathbb{R}$ and $pr + \alpha + 1 = 0$, from the definition of $L^{p,\infty}(x^\alpha)$, it is not difficult to see that $\|x^r \chi_{(0,\lambda)}(x)\|_{L^{p,\infty}(x^\alpha)} = C$, for some constant $C > 0$ independent of $\lambda > 0$. Therefore,

$$\|x^r \log(1/|x|)\, \chi_{(0,1)}(x)\|_{L^{p,\infty}(x^\alpha)} \geqslant C \log(1/\lambda),$$

so that $\|x^r \log(1/|x|)\, \chi_{(0,1)}(x)\|_{L^{p,\infty}(x^\alpha)} = \infty$. Now, if the restricted weak boundedness $[b, S_n]\colon L^{p,1}(w) \to L^{p,\infty}(w)$ holds uniformly in n for each $b \in \mathrm{BMO}$, from Theorem 1 we have

$$\left\| \log \frac{1}{|x-t|} u w^{-1/2} (1-x^2)^{-1/4} \right\|_{L^{p,\infty}(w)} < \infty,$$

$$\left\| \log \frac{1}{|x-t|} v^{-1} w^{-1/2} (1-x^2)^{-1/4} \right\|_{L^{p',\infty}(w)} < \infty$$

for each $t \in [-1, 1]$, since $b(x) = \log|x-t|^{-1} \in \mathrm{BMO}$. This leads to (3), (4), (5), which proves Corollary 2 and, as a consequence, the 'only if' part of Corollary 1.

(b) Suppose now that (3), (4), (5) hold. From Lemma 3 and the fact that generalized Jacobi polynomials belong to the class $\mathscr{H}$ (if $\alpha, \beta \geqslant -\frac{1}{2}, \gamma_i \geqslant 0$) or the class $\bar{\mathscr{H}}$ (for any $\alpha, \beta > -1$, $\gamma_i \geqslant 0$), it is easy to show that the hypotheses of Theorem 2 or Theorem 3 also hold.

3. *Fourier–Bessel series*

Let us now consider the Bessel function J_α of order $\alpha > -1$ and let $\{\alpha_n\}_{n=1}^\infty$ be the increasing sequence of the zeros of J_α. The Bessel system of order α, $\{j_n^\alpha\}_{n=1}^\infty$, where

$$j_n^\alpha(x) = 2^{1/2}|J_{\alpha+1}(\alpha_n)|^{-1} J_\alpha(\alpha_n x) \quad \text{for } n \geqslant 1,$$

is orthogonal and complete in $L^2((0,1), x\,dx)$. Let $S_n^\alpha f$ denote the nth partial sum operators

$$S_n^\alpha f(x) = \sum_{k=1}^n c_k j_k^\alpha(x), \quad c_k = c_k(f) = \int_0^1 j_k^\alpha(y) f(y)\, y\, dy.$$

THEOREM 1′. *Let U, V be two weights on* $(0,1)$. *If there exists some constant $C > 0$ such that*

$$\|U[b, S_n^\alpha](V^{-1}f)\|_{L^{p,r}(x\,dx)} \leqslant C\|f\|_{L^{q,s}(x\,dx)}$$

for each $n \geqslant 0$, $f \in L^{q,s}(x\,dx)$ (where $1 < p < \infty$, $1 < q < \infty$; either $r = p$ or $r = \infty$; either $s = q$ or $s = 1$), then

$$\left\|\log\frac{1}{|x-a|}Ux^{-1/2}\right\|_{L^{p,r}(x\,dx)} < \infty, \quad \left\|\log\frac{1}{|x-a|}V^{-1}x^{-1/2}\right\|_{L^{q',s'}(x\,dx)} < \infty$$

for each $a \in [-1, 1]$.

The proof is similar to that of Theorem 1, if we replace the previously mentioned results of **[17]** and **[13]** by the analogous results for Fourier–Bessel series (see **[10**, Lemma 2; **11**, proof of Theorem 3]).

In a similar way to the case of weights in the class $\mathscr{H}$ we obtain the following.

THEOREM 2′. *Let $1 < p < \infty$, $\alpha \geqslant -\frac{1}{2}$, let U and V be weights on* $(0,1)$ *and $b \in$ BMO. If $(x^{1-p/2}U(x)^p, x^{1-p/2}V(x)^p) \in A_p^\delta(0,1)$ for some $\delta > 1$ (where $\delta = 1$ if $u = v$), then the commutator $[b, S_n^\alpha]$ is bounded from $L^p(V^p x)$ into $L^p(U^p x)$.*

This can be proved in a similar way to Theorem 2, using **[11**, Proposition 1] instead of Lemma 1. Also, for $-1 < \alpha < -\frac{1}{2}$ a result analogous to Theorem 3 can be stated. Finally, Theorems 1′ and 2′ give the following result.

COROLLARY. *Let $1 < p < \infty$, $\alpha \geqslant -\frac{1}{2}$, and*

$$U(x) = x^a(1-x)^b \prod_{k=1}^m |x - x_k|^{b_k} \quad \textit{for } a, b, b_k \in \mathbb{R}.$$

Then the following conditions are equivalent:

(a) $\|US_n^\alpha(U^{-1}f)\|_{L^p(x\,dx)} \leqslant C\|f\|_{L^p(x\,dx)}$ *for each $f \in L^p(x\,dx)$;*
(b) $\|US_n^\alpha(U^{-1}f)\|_{L^{p,\infty}(x\,dx)} \leqslant C\|f\|_{L^p(x\,dx)}$ *for each $f \in L^p(x\,dx)$;*
(c) $\|US_n^\alpha(U^{-1}f)\|_{L^{p,\infty}(x\,dx)} \leqslant C\|f\|_{L^{p,1}(x\,dx)}$ *for each $f \in L^{p,1}(x\,dx)$;*
(d) $|1/p + \frac{1}{2}(a-1)| < \frac{1}{4}$, $-1 < pb < p-1$, $-1 < pb_k < p-1$ for $1 \leqslant k \leqslant m$.

References

1. R. Askey and S. Wainger, 'Mean convergence of expansions in Laguerre and Hermite series', *Amer. J. Math.* 87 (1965) 695–708.
2. V. M. Badkov, 'Convergence in the mean and almost everywhere of Fourier series in polynomials orthogonal on an interval', *Math. USSR-Sb.* 24 (1974) 223–256.
3. A. I. Benedek and R. Panzone, 'On mean convergence of Fourier–Bessel of negative order', *Stud. Appl. Math.* 50 (1971) 281–292.
4. A. I. Benedek and R. Panzone, 'Mean convergence of series of Bessel functions', *Rev. Un. Mat. Arg.* 26 (1972) 42–61.
5. S. Bloom, 'A commutator theorem and weighted BMO', *Trans. Amer. Math. Soc.* 292 (1985) 103–122.
6. R. R. Coifman and M. A. M. Murray, 'Uniform analyticity of orthogonal projections', *Trans. Amer. Math. Soc.* 312 (1989) 779–817.
7. R. R. Coifman, R. Rochberg and G. Weiss, 'Factorization theorems for Hardy spaces in several variables', *Ann. of Math.* 103 (1976) 611–635.
8. J. García Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics* (North-Holland, Amsterdam, 1985).
9. J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, 'Weighted L^p-boundedness of Fourier series with respect to generalized Jacobi weights', *Publ. Mat.* 35 (1991) 449–459.
10. J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, 'Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal systems', *Proc. Amer. Math. Soc.* 116 (1992) 457–464.
11. J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, 'Mean and weak convergence of Fourier–Bessel series', *J. Math. Anal. Appl.* 173 (1993) 370–389.
12. J. J. Guadalupe, M. Pérez and J. L. Varona, 'Mean and weak convergence of some orthogonal Fourier expansions by using A_p theory', *Proceedings of the International Congress, Laredo, Spain,* 1987, Lecture Notes in Pure and Applied Mathematics 117 (ed. J. Vinuesa; Marcel Dekker, New York, 1989) 161–169.
13. J. J. Guadalupe, M. Pérez and J. L. Varona, 'Weighted weak behaviour of Fourier–Jacobi series', *Math. Nachr.* 158 (1992) 161–174.
14. T. Kato, *Perturbation theory for linear operators* (Springer, Berlin, 1976).
15. E. Laeng, 'Analytic dependence of orthogonal polynomials', *Rev. Mat. Iberoamericana* 7 (1991) 287–312.
16. A. Máté, P. Nevai and V. Totik, 'Asymptotics for the ratio of the leading coefficients of orthogonal polynomials on the unit circle', *Constr. Approx.* 1 (1985) 63–69.
17. A. Máté, P. Nevai and V. Totik, 'Necessary conditions for weighted mean convergence of Fourier series in orthogonal polynomials', *J. Approx. Theory* 46 (1986) 314–322.
18. B. Muckenhoupt, 'Mean convergence of Jacobi series', *Proc. Amer. Math. Soc.* 23 (1969) 306–310.
19. B. Muckenhoupt, 'Mean convergence of Hermite and Laguerre series, II', *Trans. Amer. Math. Soc.* 147 (1970) 433–460.
20. C. J. Neugebauer, 'Inserting A_p weights', *Proc. Amer. Math. Soc.* 87 (1983) 644–648.
21. H. Pollard, 'The mean convergence of orthogonal series, II', *Trans. Amer. Math. Soc.* 63 (1948) 355–367.
22. H. Pollard, 'The mean convergence of orthogonal series, III', *Duke Math. J.* 16 (1949) 189–191.
23. E. A. Rahmanov, 'On the asymptotics of the ratio of orthogonal polynomials, II', *Math. USSR.-Sb.* 46 (1983) 105–117.
24. C. Segovia and J. L. Torrea, 'Vector-valued commutators and applications', *Indiana Univ. Math. J.* 38 (1989) 959–971.
25. C. Segovia and J. L. Torrea, 'Weighted inequalities for commutators of fractional and singular integrals', *Publ. Mat.* 35 (1991) 209–235.
26. C. Segovia and J. L. Torrea, 'Higher order commutators for vector-valued Calderón–Zygmund operators', *Trans. Amer. Math. Soc.* 336 (1993) 537–556.
27. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces* (Princeton University Press, 1975).
28. G. M. Wing, 'The mean convergence of orthogonal series', *Amer. J. Math.* 72 (1950) 792–808.

J.J.G.
Departamento de Matemáticas y Computación
Universidad de La Rioja
26004 Logroño
Spain

M.P. and F.J.R.
Departamento de Matemáticas
Universidad de Zaragoza
50009 Zaragoza
Spain

E-mail address (M.P.): mperez@posta.unizar.es