Bull. London Math. Soc. 39 (2007) 203–208 © 2007 London Mathematical Society
doi:10.1112/blms/bdl027

DIMENSION DEPENDENCY OF THE WEAK TYPE $(1, 1)$ BOUNDS FOR MAXIMAL FUNCTIONS ASSOCIATED TO FINITE RADIAL MEASURES

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Abstract

We show that the smallest constants appearing in the weak type (1,1) inequalities satisfied by the centred Hardy–Littlewood maximal function associated to some finite radial measures, such as the standard gaussian measure, grow exponentially fast with the dimension.

1. *Introduction*

Let M be the centred maximal operator (cf. equation (2.0.1) below) associated to euclidean balls and Lebesgue measure. It is well known that if $1 < p \leqslant \infty$, then there exists a constant c_p such that for all $f \in L^p(\mathbb{R}^d)$, we have $\|Mf\|_p \leqslant c_p \|f\|_p$. When $p = \infty$, trivially $c_p = 1$. The standard proof of $\|Mf\|_p \leqslant c_p \|f\|_p$ for $1 < p < \infty$, via weak type (1,1) inequalities and interpolation, rests on covering lemmas of a geometric character. This fact leads to values of c_p that grow exponentially with the dimension d. An alternative proof using the method of rotations gives constants c_p whose growth is linear in d (but yields no weak type (1,1) inequality). In this context, E. M. Stein was able to show that in fact one can take c_p to be independent of d [**11**, **12**]; see also [**13**]. A motivation for the study of L^p bounds uniform in d comes from the desire to extend (at least some parts of) harmonic analysis in $\mathbb{R}^d$ to the infinite dimensional case. Stein's result was generalized to the maximal function defined using an arbitrary norm by J. Bourgain [**4**–**6**] and A. Carbery [**7**] when $p > 3/2$. For ℓ_q balls, $1 \leqslant q < \infty$, D. Müller [**10**] showed that uniform bounds again hold for every $p > 1$ (given $1 \leqslant q < \infty$, the ℓ_q balls are defined using the norm $\|x\|_q := (x_1^q + x_2^q + \cdots + x_d^q)^{1/q}$). With respect to weak type bounds, in [**14**] E. M. Stein and J. O. Strömberg proved, among other things, that the smallest (that is, the best) constants in the weak type (1,1) inequality satisfied by M grow at most like $O(d)$, and asked if uniform bounds could be found. Since then, there has been remarkably little progress on this question (see, for instance, [**1**, **2**] for the case of cubes).

Here we study the weak type (1,1) problem for integrable radial densities defined via bounded decreasing functions, the canonical example being the standard gaussian measure. This is a natural variant of Stein and Strömberg's question, given the growing interest in what has been termed 'gaussian harmonic analysis', where Lebesgue measure is replaced by the standard gaussian measure, and also because of the importance of gaussian measures and other probabilities in the infinite dimensional setting (see, for instance, [**3**]). For the measures considered in this paper, instead of uniform bounds we have exponential increase: if μ is a finite radial Borel measure on $\mathbb{R}^d$ defined by a bounded decreasing function f, and if c_d denotes the smallest constant appearing in the weak type (1,1) inequality satisfied by the associated

Received 18 July 2005; revised 10 May 2006; published online 13 February 2007.
2000 *Mathematics Subject Classification* 42B25.
Partially supported by Grant BFM2003-06335-C03-03 of the D.G.I. of Spain.

maximal function M_μ, then for every d we have

$$c_d \geqslant \left(1 + \frac{2\sqrt{2}}{\sqrt{3\pi d}}\sqrt{1+\frac{1}{d}}\right)^{-1}\left(\frac{2}{\sqrt{3}}\right)^{d/6}.$$

2. *Notation and results*

Given a locally finite Borel measure μ on $\mathbb{R}^d$ (so compact sets have finite measure) and a locally integrable function f, the associated centred maximal function $M_\mu f$ is defined by

$$M_\mu f(x) := \sup_{\{r>0:\mu(B(x,r))>0\}} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f| d\mu, \tag{2.0.1}$$

where $B(x,r)$ denotes the euclidean *closed* ball of radius $r>0$ centred at x (the choice of closed balls in the definition is mere convenience; using open balls instead does not change the value of $M_\mu f(x)$). The boundary of $B(x,r)$ is the sphere $\mathbb{S}(x,r)$. Sometimes we use $B^d(x,r)$ and $\mathbb{S}^{d-1}(x,r)$ to make their dimensions explicit. If $x=0$ and $r=1$, we just write B^d and $\mathbb{S}^{d-1}$.

It is a consequence of the Besicovitch covering theorem that there exists a constant $c=c(d)$, independent of μ, such that for every $f\in L^1(\mathbb{R}^d,\mu)$ and every $\alpha>0$, we have $\alpha\mu(\{M_\mu f \geqslant \alpha\}) \leqslant c\|f\|_1$. In fact, by the theorem in [**15**, p. 227] we may take $c=(2.641+o(1))^d$. Of course, if we are interested in the smallest such c, then $c=c(\mu)$ will depend on μ. Note that it makes no difference in the determination of the smallest constant if instead of the strict inequality $\{M_\mu f>\alpha\}$ we use $\{M_\mu f\geqslant\alpha\}$.

The following easy result highlights the fact that uniform bounds will often fail to exist when dealing with sequences $\{\mu_d\}_{d=1}^\infty$ of measures satisfying $\mu_d(\mathbb{R}^d)<\infty$, due to the decay this condition imposes. We take $a/\mu(B(y,r))$ to mean ∞ when $a>0$ and $\mu(B(y,r))=0$.

PROPOSITION 2.1. *Let μ be a locally finite Borel measure on $\mathbb{R}^d$, and let c_d be the smallest constant appearing in the weak type* (1,1) *inequality satisfied by M_μ. Given any ball $B(x,r)$ with x in the support of μ,*

$$c_d \geqslant \inf_{y\in\mathbb{S}(x,r)} \frac{\mu(B(x,r))}{\mu(B(y,r))}.$$

Proof. Fix $r>0$ and let x belong to the support of μ. By a standard approximation argument we may consider, instead of a function, the Dirac delta δ_x placed at x. Thus $c_d \geqslant \sup_{\alpha>0}\alpha\mu(\{M_\mu\delta_x\geqslant\alpha\})$. Note that for some $y\in\mathbb{S}(x,r)$, $\mu(B(y,r))>0$. Also, if $y\in\mathbb{S}(x,r)$ and $\mu(B(y,r))=0$, then $M\delta_x(y)=\infty$, since for all $s>r$, $\mu(B(y,s))>0$ and $\lim_{s\downarrow r}\mu(B(y,s))=0$. Let $\alpha_0 := \inf_{y\in\mathbb{S}(x,r)} 1/\mu(B(y,r))$. Then $M_\mu\delta_x(z)\geqslant\alpha_0$ for every $z=(1-t)x+ty$, $y\in\mathbb{S}(x,r)$, $0<t\leqslant 1$, since $x\in B(z,tr)\subset B(y,r)$. It follows that $B(x,r)\subset\{M_\mu\delta_x\geqslant\alpha_0\}$, so $c_d\geqslant \alpha_0\mu(B(x,r))$, as claimed. □

REMARK 2.2. Let μ be a locally finite Borel measure on $\mathbb{R}^d$. By the Lebesgue theorem on differentiation of integrals, $M_\mu f(x)\geqslant |f|(x)$ for μ a.e. (almost everywhere) x. Now, fix $\varepsilon>0$, and let $f:=(1+\varepsilon)\chi_{B(0,1)}$. Then $\mu(B(0,1))=\mu(\{f>1\})\leqslant(1+\varepsilon)c_d\int\chi_{B(0,1)}d\mu$, so $c_d\geqslant 1$.

Fix $d\in\mathbb{N}\setminus 0$. Let $f:[0,\infty)\to[0,\infty)$ be a non-increasing function, let σ^{d-1} denote the area on the unit sphere $\mathbb{S}^{d-1}$, let $\tilde\sigma^{d-1}$ denote the *normalized* area on $\mathbb{S}^{d-1}$ (thus $\tilde\sigma^{d-1}$ is a probability measure), and let λ^d be Lebesgue measure on $\mathbb{R}^d$. Then the function f defines a rotationally invariant (or radial) measure μ via

$$\mu(A) := \int_A f(|y|)d\lambda^d(y). \tag{2.2.1}$$

We remark that since f is non-increasing, it is bounded by $f(0)$. Additionally, we shall assume that f is not 0 a.e., so $\mu(\mathbb{R}^d) > 0$, and furthermore, that $f(x)x^{d-1} \in L^1([0,\infty))$, so $\mu(\mathbb{R}^d) < \infty$, as can be seen by integrating in polar coordinates.

THEOREM 2.3. *Fix $d \in \mathbb{N} \setminus 0$. Let $f : [0,\infty) \to [0,\infty)$ be a non-increasing function and let μ be the radial measure defined via (2.2.1). Assume f is such that $0 < \mu(\mathbb{R}^d) < \infty$, and let c_d be the smallest constant appearing in the weak type (1,1) inequality satisfied by M_μ. Then*

$$c_d \geqslant \left(1 + \frac{2\sqrt{2}}{\sqrt{3\pi d}}\sqrt{1+\frac{1}{d}}\right)^{-1} \left(\frac{2}{\sqrt{3}}\right)^{d/6}. \tag{2.3.1}$$

Proof. Since $c_1 \geqslant 1$ by Remark 2.2, the lower bound (2.3.1) holds for $d = 1$, so we may assume that $d \geqslant 2$. Given a unit vector $v \in \mathbb{R}^d$ and $\varepsilon \in [0,1)$, the ε spherical cap about v is the set $C(\varepsilon, v) := \{\theta \in \mathbb{S}^{d-1} : \langle \theta, v\rangle \geqslant \varepsilon\}$. Note that spherical caps are just geodesic balls $B_{\mathbb{S}^{d-1}}(x,r)$ in $\mathbb{S}^{d-1}$. In the special case $v = e_1 = (1,0,\ldots,0)$, $\varepsilon = 2^{-1}$, we have $C(2^{-1}, e_1) = B_{\mathbb{S}^{d-1}}(e_1, \pi/3)$. Next we remind the reader of some well-known facts that will be used in the sequel: (i) $\lambda^d(B^d) = \pi^{d/2}/\Gamma(1+d/2)$; (ii) $\sigma^{d-1}(\mathbb{S}^{d-1}) = d\lambda^d(B^d)$; (iii) $\sigma^{d-1}(B_{\mathbb{S}^{d-1}}(x,r)) = \sigma^{d-2}(\mathbb{S}^{d-2})\int_0^r \sin^{d-2} t\, dt$ (see, for instance, [**9**, (A.11), p. 259] for a more general statement). We shall also use the following known and elementary estimate (cf. [**16**, Exercise 5, p. 216]). The short derivation is included for the reader's convenience. Recall that on $\{x > 0\}$, Γ is log-convex and $\Gamma(x+1) = x\Gamma(x)$. Writing $d + 2 = 2^{-1}(d+1) + 2^{-1}(d+3)$, we have

$$\frac{\Gamma(1+d/2)}{\Gamma(1/2+d/2)} \leqslant \frac{(\Gamma(2^{-1}(d+1)))^{1/2}(\Gamma(2^{-1}(d+3)))^{1/2}}{\Gamma(2^{-1}(d+1))} = \left(\frac{d+1}{2}\right)^{1/2}. \tag{2.3.2}$$

Note that from (i), (ii), (iii), (2.3.2) and the fact that $\cos t \geqslant 1/2$ on $[0,\pi/3]$, we get the following upper bound on the normalized area of $C(2^{-1}, e_1)$:

$$\tilde{\sigma}^{d-1}(C(2^{-1}, e_1)) \leqslant 2\frac{\sigma^{d-2}(\mathbb{S}^{d-2})}{\sigma^{d-1}(\mathbb{S}^{d-1})}\int_0^{\pi/3} \sin^{d-2} t \cos t\, dt \tag{2.3.3}$$

$$= \frac{2}{d}\frac{\lambda^{d-1}(B^{d-1})}{\lambda^d(B^d)}\left(\frac{\sqrt{3}}{2}\right)^{d-1} \leqslant \left(\frac{\sqrt{3}}{2}\right)^{d}\frac{2\sqrt{2}}{\sqrt{3\pi d}}\sqrt{1+\frac{1}{d}}. \tag{2.3.4}$$

Note next that the function $h(R) := \mu(B(0,R))/\mu(B(0,(\sqrt{3}/2)R))$ is continuous, and $\lim_{R\to\infty} h(R) = 1$ by the finiteness of the measure. It follows that there is a largest real number R_1 such that $h(R_1) = (2/\sqrt{3})^{d/6}$, provided of course that the set $\{h \geqslant (2/\sqrt{3})^{d/6}\}$ is non-empty. To see that this is always the case, note that since f is non-increasing, the same happens with the averages $(1/\lambda^d(B(0,R)))\int_{B(0,R)} f(|x|)dx$. Thus $\lim_{R\to 0}(1/\lambda^d(B(0,R)))\int_{B(0,R)} f(|x|)dx = L$ exists and $L \leqslant f(0) < \infty$. It follows that

$$\lim_{R\to 0} h(R) = \lim_{R\to 0}\frac{(\lambda^d(B(0,R))/\lambda^d(B(0,R)))\int_{B(0,R)} f(|x|)dx}{(\lambda^d(B(0,(\sqrt{3}/2)R))/\lambda^d(B(0,(\sqrt{3}/2)R)))\int_{B(0,(\sqrt{3}/2)R)} f(|x|)dx} = (2/\sqrt{3})^d.$$

By rotational invariance and the previous proposition, in order to prove the theorem it is enough to check that

$$\frac{\mu(B(0,R_1))}{\mu(B(R_1e_1,R_1))} \geqslant \left(1 + \frac{2\sqrt{2}}{\sqrt{3\pi d}}\sqrt{1+\frac{1}{d}}\right)^{-1}\left(\frac{2}{\sqrt{3}}\right)^{d/6}.$$

We split $\mu(B(R_1e_1,R_1)) = \mu(B(0,R_1)\cap B(R_1e_1,R_1)) + \mu(B(0,R_1)^c \cap B(R_1e_1,R_1))$ and estimate each of the summands. Note that

$$B(0,R_1)\cap B(R_1e_1,R_1) \subset B(2^{-1}R_1e_1, 2^{-1}\sqrt{3}R_1).$$

For every pair of points (x, y) with $x \in B(0, R_1) \setminus B(R_1e_1, R_1)$, $y \in B(R_1e_1, R_1) \setminus B(0, R_1)$, we have $|x| < |y|$, so $f(|x|) \geqslant f(|y|)$. By the choice of R_1 and the preceding observation,

$$\begin{aligned}\mu(B(0,R_1)\cap B(R_1e_1,R_1)) &\leqslant \mu(B(2^{-1}R_1e_1, 2^{-1}\sqrt{3}R_1)) \\ &\leqslant \mu(B(0,2^{-1}\sqrt{3}R_1)) = (\sqrt{3}/2)^{d/6}\mu(B(0,R_1)).\end{aligned} \tag{2.3.5}$$

Let E be the semi-cone in $\mathbb{R}^d$ defined by $x_1 = 3^{-1/2}\sqrt{x_2^2 + \cdots + x_d^2}$, and let $E' := \{x_1 \geqslant 3^{-1/2}\sqrt{x_2^2 + \cdots + x_d^2}\}$ denote the solid semi-cone determined by E. Then

$$B(0,R_1)^c \cap B(R_1e_1,R_1) \subset B(0,(2/\sqrt{3})^5R_1)\cap E'.$$

By rotational invariance of the probability measure

$$\nu(A) := \frac{\mu(B(0,(2/\sqrt{3})^5R_1)\cap A)}{\mu(B(0,(2/\sqrt{3})^5R_1))},$$

the ν-measure of E' is just the normalized area of its intersection with the sphere, that is

$$\nu(E') = \tilde{\sigma}^{d-1}(E'\cap\mathbb{S}^{d-1}) = \tilde{\sigma}^{d-1}(C(2^{-1},e_1)).$$

From the choice of R_1 and the upper bound (2.3.3), (2.3.4) on $\tilde{\sigma}^{d-1}(C(2^{-1},e_1))$, we get

$$\begin{aligned}\mu(B(0,R_1)^c\cap B(R_1e_1,R_1)) &\leqslant \nu(E')\mu(B(0,(2/\sqrt{3})^5R_1)) \\ &< \tilde{\sigma}^{d-1}(C(2^{-1},e_1))(2/\sqrt{3})^{5d/6}\mu(B(0,R_1)) \\ &\leqslant \left(\frac{2}{\sqrt{3}}\right)^{5d/6}\mu(B(0,R_1))\left(\frac{\sqrt{3}}{2}\right)^{d}\frac{2\sqrt{2}}{\sqrt{3\pi d}}\sqrt{1+\frac{1}{d}} \\ &= \mu(B(0,R_1))\left(\frac{\sqrt{3}}{2}\right)^{d/6}\frac{2\sqrt{2}}{\sqrt{3\pi d}}\sqrt{1+\frac{1}{d}}.\end{aligned} \tag{2.3.6}$$

Putting together the estimates starting at (2.3.5) and at (2.3.6), we obtain

$$\frac{\mu(B(0,R_1))}{\mu(B(R_1e_1,R_1))} \geqslant \frac{\mu(B(0,R_1))}{\left(1+\frac{2\sqrt{2}}{\sqrt{3\pi d}}\sqrt{1+\frac{1}{d}}\right)\left(\frac{\sqrt{3}}{2}\right)^{d/6}\mu_d B(0,R_1)},$$

as desired. □

Remark 2.4. For each $d \in \mathbb{N}\setminus 0$ let $f_d : [0,\infty) \to [0,\infty)$ be a non-increasing integrable function, not 0 a.e., such that $f_d(x)x^{d-1}$ is integrable, and let μ_d be the nontrivial finite radial measure on $\mathbb{R}^d$ defined using f_d. By Theorem 2.3, there cannot be a uniform bound for the smallest constants appearing in the weak type (1,1) inequalities satisfied by μ_d. Note that for different values of d the functions f_d might be totally unrelated, or on the other extreme, might always be the same function f (provided $f(x)x^{d-1}$ is integrable for all d), but this has no effect on the lack of uniform bounds.

Remark 2.5. When $d = 1$, one can easily improve on the trivial bound $c_1 \geqslant 1$. Fix $\varepsilon > 0$ and choose $R > 0$ such that $\mu([0,2R])/\mu([0,R]) < 1+\varepsilon$. Then

$$\frac{\mu([-R,R])}{\mu([0,2R])} = \frac{\mu([-R,R])}{\mu([-2R,0])} > \frac{2\mu([0,R])}{(1+\varepsilon)\mu([0,R])} = \frac{2}{1+\varepsilon},$$

so in fact $c_1 \geqslant 2$ for all finite nontrivial radial measures on $\mathbb{R}$.

Remark 2.6. We briefly comment on the role that different assumptions play in the proof of Theorem 2.3. Finiteness of μ was only used to show that there is an $R > 0$ such that $h(R) \geqslant (2/\sqrt{3})^{d/6}$ and $h((2/\sqrt{3})^iR) \leqslant (2/\sqrt{3})^{d/6}$ for $i = 1,2,3,4,5$. Thus exponential dependency holds for families of radial measures, not necessarily finite, provided there exists a ball in each dimension with centre in the support of the corresponding measure, and a similar kind of decay (clearly the proof can be adapted to decays lower than the one considered above). Normalization of the measures is not an issue either: if $c > 0$, then $M_\mu = M_{c\mu}$. The radial assumption makes it easy to check the decay and to apply Proposition 2.1, but as noted above, the existence of some ball with such decay is likely to be prevalent even among nonradial (finite) measures. Regarding the assumption of absolute continuity of the measures μ_d in $\mathbb{R}^d$, it is a simple way to ensure that in some sense the dimension of μ_d goes to infinity as $d \to \infty$ (cf. for instance [**8**, Chapter 10] for a definition of dimension of a measure, and in particular, [**8**, Proposition 10.5, p. 174]: it follows that if μ_d is absolutely continuous in $\mathbb{R}^d$ then $\dim \mu_d = d$). A simple example where $\dim \mu_d$ does not grow with d is obtained by setting $\mu_d = \delta_0$ for every d. Here we always have $\dim \mu_d = 0$ and $c_d = 1$. It is also easy to obtain singular examples where there is dependency of c_d on d: denote respectively by γ_d and by δ_0 the standard gaussian measure on $\mathbb{R}^d$ and the point mass at the origin of $\mathbb{R}^d$. Then $\gamma_d \times \delta_0$ on $\mathbb{R}^{2d}$ is singular, d-dimensional, and c_d grows exponentially with d.

Finally, we point out that there are some natural families of singular radial finite measures to which the arguments in the proof of Theorem 2.3 do not apply, such as, for instance, area on $\mathbb{S}^{d-1}$.

REMARK 2.7. When dealing with concrete families of measures, the additional information may lead to more precise bounds. For instance, let $\nu_d(A) := \lambda^d(A \cap B^d)$ be Lebesgue measure restricted to the unit ball in $\mathbb{R}^d$. By Remark 2.5, $c_1(\nu_1) \geqslant 2$, and for $d \geqslant 2$, the argument used in (2.3.5) immediately gives $c_d(\nu_d) \geqslant (2/\sqrt{3})^d$. Nevertheless, it is possible to do better by estimating directly the volume of the solid cap $B^d \cap \{x_1 \geqslant 2^{-1}\}$:

$$\nu_d(B(e_1,1)) = 2\lambda^d(B^d \cap \{x_1 \geqslant 2^{-1}\}) = 2\lambda^{d-1}(B^{d-1}) \int_{1/2}^{1} \left(\sqrt{1-x_1^2}\right)^{d-1} dx_1$$
$$\leqslant 4\lambda^{d-1}(B^{d-1}) \int_{\pi/6}^{\pi/2} \cos^d t \sin t dt = \frac{4}{d+1}\left(\frac{\sqrt{3}}{2}\right)^{d+1} \lambda^{d-1}(B^{d-1}).$$

Using Proposition 2.1 and (2.3.2) we have

$$c_d(\nu_d) \geqslant \frac{\lambda^d(B^d)}{\nu_d(B(e_1,1))} \geqslant \frac{\sqrt{\pi(d+1)}}{\sqrt{6}}\left(\frac{2}{\sqrt{3}}\right)^d. \tag{2.7.1}$$

The same estimate holds if $R > 0$ is chosen arbitrarily and we define $\nu_d(A) := \lambda^d(A \cap B(0,R))$, since $\lambda^d(B(0,R))/\nu_d(B(Re_1,R))$ is independent of R. Now fix d. As noted in Remark 2.6, multiplying ν_d by a nonzero constant does not change any estimate; set $\nu_n(A) := C_n\lambda^d(A \cap B(0,1/n))$, where C_n is chosen to make ν_n a probability measure. Then $\nu_n \to \delta_0$ in the weak* topology (weakly, in the standard probabilistic terminology), but by Remark 2.5, the lower bound (2.7.1) and the preceding comments (or by Theorem 2.3 if d is high enough), $\liminf_n c_d(\nu_n) > 1 = c_d(\delta_0)$. A similar observation can be made about the sequence defined by $\nu_n(A) := \lambda^d(A \cap B(0,n))$ and λ^d.

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