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ORIGINAL ARTICLE

A note on lower bounds for relative equilibria in the main problem of artificial satellite theory

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Abstract In the analytical approach to the main problem in satellite theory, the consideration of the physical parameters imposes a lower bound for normalized Hamiltonian. We show that there is no elliptic frozen orbits, at critical inclination, when we consider small values of H, the third component of the angular momentum. The argument used suggests that it might be applied also to more realistic zonal and tesseral models. Moreover, for almost polar orbits, when H may be taken as another small parameter, a different approach that will simplify the ephemerides generators is proposed.

Keywords Satellite theory · Main problem · Normalization · Critical inclination · Frozen orbits · Very eccentric orbits

1 Introduction

Deprit reporting to the Commission 7 of IAU at the XIX General Assembly in Delhi (Deprit 1985), and in a satellite Workshop (Coffey et al. 1986a), presented the answer to the problem of the critical inclination in satellite theory. Considering Brouwer's Hamiltonian, the second order truncation of the normalized system, they had found that two pitchfork bifurcations, stemming from circular orbits, relate with the families of orbits with stationary perigee in

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the averaged main problem of the satellite theory; this connection elucidates the nature of the *critical inclination*. The issue was fully documented in Coffey et al. (1986b), which we refer to as the CDM paper (See also Cushman 1983, 1988, 1991).

Mainly concerned with the qualitative dynamics of almost circular orbits, an open problem since the early work of Orlov, and after resolving it, CDM left unfinished the study of the "evolution" of the stable and unstable orbits of stationary perigee with the variation of the integral H, as well as the study of the possible existence of other solutions. Indeed, formulas 24_1 and 24_2 in CDM paper, based on Newton–Raphson method, are local and lose their meaning when H becomes small (the fact that H is 'hidden' in $q = H^2/\mu$ may explain why this has passed unnoticed). The same ought to be said in relation to the work of Cushman (1983, 1991) on this problem; in his analysis the only lower limit for $|H|$ is to be positive. In a recent publication of Chang and Marsden (2003) this issue still passes without any warning. To our knowledge what we communicate has not been published before.

In this note we point out that, in our view, research done both in the zonal and tesseral approach to satellite theory requires reconsideration, taking into account: (i) the radius of the planet, (ii) the values of the third component of the angular momentum and, (iii) the domain of convergence of the normalized Hamiltonian of the problem used in those studies. In order to make the note shorter we refer our comments to the main problem. The paper is organized as follows. In Sect. 2 we first fix notation presenting the Hamiltonian in polar-nodal variables. Moreover, we refer briefly to the normalized Hamiltonian including explicitly the Coffey–Deprit model, the third order truncation in closed form of the eccentricity. It serves to identify a rough lower bound estimate for G, the norm of the angular momentum, for the validity of the normalized Hamiltonian. In Sect. 4.1, we come back to this Hamiltonian function.

In Sect. 3 we refer to the relative equilibria. The previous mentioned papers paid no attention in their studies to the role played by α, the equatorial 'radius' of the planet, one of the physical parameters of the problem. It is worth mentioning that collision orbits had been already taken into account by Hough (1981). First we gather some basic expressions for non-impacting and grazing orbits in the bounded 'zero order satellite problem' (an infinitesimal point mass around an oblate planet) basic for astrodynamics implications. Then, we show that first order normalization and the search for relative equilibria, gives already a lower bound for H, apart from the relation between inclination and eccentricity for frozen orbits (critical inclination). As an astrodynamics limit we refer also to *frozen-grazing* orbits. The second truncation (Brouwer model) is essential in order to identify two pairs of stable and unstable elliptic frozen orbits bifurcating from circular orbits, as we have mentioned above (CDM paper). In particular, our point in this note is that the claim of CDM (p. 387) about the relative equilibria as function of H has to be modified in the sense that there are no relative equilibria when H is small (see Sect. 4). But having said this, it has to be stressed that CDM expressions are valid for relative equilibria corresponding to non-impacting orbits.

In Sect. 4, we consider the case when H is small. First the Coffey–Deprit Hamiltonian is used. Taking the model by itself, we find 'new relative equilibria' $(\tilde{G}, \tilde{g})$. Nevertheless, we see that $\tilde{G}^4 \approx J_2$, *i.e.*, they are out of the domain of convergence, according to the lower bound identified in Sect. 2. In other words, those solutions belong to the truncated model but not to the averaged system. In Sect. 4.2 we briefly approach the problem from a different angle. For the case of the family of almost polar orbits (say $|H| \leq \sqrt{J_2}$), we propose to treat the problem considering the perturbation split in first and higher order terms. This lead us to the conclusion that the only relative equilibria are circular orbits.

Some related questions, like comparing with numerical studies of Broucke (1994), are not tackled in this note. Indeed, he considered large values of $|J_2| = 0.2$ and reported that the family of almost polar orbits seems to maintain the pattern coming from the critical inclination. This, as well as the persistence or not of our claim when other harmonics coefficients are taken into account, is now under study.

A final comment is due here. Perturbed Keplerian systems with an axial symmetry, after been normalized, have to be reduced in order to bring the system finally to the twice reduced space. All this was explained in detail by Cushman using regular and singular reduction theory. However, in this note we do not need to make use of the proper coordinates (invariants) of this space. For the evolution of the elliptic relative equilibria the local chart defined by the Delaunay variables (g, G) is sufficient.

2 On the main problem and its normalization

In Cartesian coordinates $\mathbf{x}$, $\mathbf{X}$ the Hamiltonian function of the so called *main problem* of the artificial satellite theory (see Brouwer 1959) is usually given by

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(\mathbf{x}, \mathbf{X}; \mu, \alpha, J_2)\\ &= \frac{1}{2}(X^2 + Y^2 + Z^2) - \frac{\mu}{r} + J_2\,\frac{1}{2}\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2\left(3\,\frac{z^2}{r^2} - 1\right),\end{aligned} \tag{1}$$

and the corresponding differential system defined in $\Omega \subset (\mathbf{R}^3 - \{\mathbf{0}\}) \times \mathbf{R}^3$, where $r = \sqrt{x^2 + y^2 + z^2}$, μ is the gravitational constant, α is its equatorial radius and J_2 is the oblateness coefficient of the planet (a small positive quantity). Thus, although the non-Keplerian part is factorized by the quantity $J_2\,\mu\,\alpha^2$, we prefer to maintain explicit their presence.

Apart from the Hamiltonian function itself which is an integral, the Hamiltonian system defined by (1) has an axial symmetry, which is made manifest immediately if we use, for instance, cylindrical coordinates. In other words, the third component N of the *angular momentum* vector $\mathbf{x} \times \mathbf{X}$ is a second integral (later in the note we write $N = H$). Although considerable effort has been put in searching for a possible third integral, so far only the case $\Theta = \|\mathbf{x} \times \mathbf{X}\| = |N|$, called *equatorial main problem*, reduces to a 1-DOF system. Thus, our Hamiltonian function defines generically a 2-DOF system. In the open domain $\mathcal{I} = \{(\mathbf{x}, \mathbf{X})|\ 0 < \Theta,\ N < \Theta\} \subset \Omega$ of 'inclined orbits' (that is excluding the possibility of rectilinear and equatorial trajectories), the Hamiltonian function, written in polar-nodal variables, $(r, \theta, \nu, R, \Theta, N)$, is given (see Deprit 1981) by

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - J_2\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2\frac{1}{2}(1 - 3s^2\,\sin^2\theta), \tag{2}$$

where $s = \sqrt{1 - (N/\Theta)^2}$, and ν is a cyclic variable. Our interest focuses on the restrictions to the problem associated with α and J_2. More precisely, α relates to conditions for bounded non impacting orbits, i.e., characterizing the set of bounded orbits where $r \geq \alpha$. The influence of J_2 is related to the validity of convergence and relative equilibria of the normal form, which is obtained as power expansion of J_2. Note that some authors, dealing with the problem with numerical methods, have considered the problem defined by the function (2), studying the influence of large values of J_2 in the model. We will restrict our considerations to the region of phase space defined by negative values of the Hamiltonian. In other words to bounded motions of elliptic perturbed type. Then, we will use 'Delaunay variables' (ℓ, g, h, L, G, H), the Hamiltonian version of the classic 'orbital elements' (see Delaunay 1867; Chang and Marsden 2003).

2.1 The Coffey–Deprit normalized Hamiltonian. A rough lower bound for convergence

Coffey and Deprit (1982) presented the normalization of the main problem in closed form as a function of the eccentricity, up to third order, a long standing open question since the time of Brouwer (1959). That achievement came after a Lie transform was found by Deprit, dubbed as *Elimination of the Parallax*, which allowed to simplify powers $(\alpha/r)^n$, with $n \geq 3$, to $(\alpha/r)^2$. More recently Healy (2000) has published a new Delaunay normalization reaching up to order 6. For this note, we do not need to consider order higher than three. From Coffey et al. (1982) we borrow

$$\mathcal{K} = \sum_{n\geq 0}^{3} \frac{\epsilon^n}{n!}\mathcal{K}_n + \mathcal{O}(\epsilon^4), \tag{3}$$

where

$$\begin{aligned} \mathcal{K}_0 &= -\frac{\mu^2}{2L^2}, \\ \mathcal{K}_i &= \eta\frac{\mu^2}{L^2}\left(\frac{\alpha^2\mu^2}{G^4}\right)^i \textstyle\sum_{0\leq 2k\leq i} P_{i,2k}(s^2,\eta)\cos 2kg \quad \text{with } i\geq 1, \end{aligned} \tag{4}$$

and $\epsilon = J_2$. The coefficients $P_{i,2k}$

$$P_{1,0} = \frac{3}{4}s^2 - \frac{1}{2},$$

$$P_{2,0} = -\left(\frac{15}{64}s^4 + \frac{3}{8}s^2 - \frac{3}{8}\right)\eta^2 - \left(\frac{27}{16}s^4 - \frac{9}{4}s^2 + \frac{3}{4}\right)\eta - \frac{105}{64}s^4 + \frac{15}{4}s^2 - \frac{15}{8},$$

$$P_{2,2} = \left(\frac{45}{32}s^4 - \frac{21}{16}s^2\right)(\eta^2 - 1),$$

$$\begin{aligned} P_{3,0} = &\left(\frac{675}{256}s^6 + \frac{315}{128}s^4 - \frac{225}{32}s^2 + \frac{45}{16}\right)\eta^3 - \left(\frac{6579}{256}s^6 - \frac{5643}{128}s^4 + \frac{711}{32}s^2 - \frac{81}{16}\right)\eta^2 \\ &+ \left(\frac{2835}{256}s^6 - \frac{4185}{128}s^4 + \frac{945}{32}s^2 - \frac{135}{16}\right)\eta + \frac{13725}{256}s^6 - \frac{14319}{128}s^4 \\ &+ \frac{2421}{32}s^2 - \frac{315}{16}, \end{aligned}$$

$$\begin{aligned} P_{3,2} = &-\left(\frac{2025}{128}s^6 - \frac{405}{16}s^4 + \frac{315}{32}s^2\right)\eta^3 - \left(\frac{675}{512}s^6 - \frac{207}{32}s^4 + \frac{9}{2}s^2\right)\eta^2 \\ &+ \left(\frac{1215}{128}s^6 - \frac{243}{16}s^4 + \frac{189}{32}s^2\right)\eta + \frac{3915}{512}s^6 - \frac{531}{32}s^4 + \frac{135}{16}s^2, \end{aligned}$$

are polynomials in $s = \sin i = \sqrt{1 - H^2/G^2}$ and $\eta = G/L$. As usual, primes in the variables have been dropped. Note that associated with the Hamiltonian we have another power series $\mathcal{W}$ defining the generator of the Lie–Deprit transformation. In fact two, related with the Elimination of the Parallax and the Delaunay normalization (for details see Coffey and Deprit (1982) and Healy (2000)).

As we know the previous expression is formal in the sense that nothing is said about its convergence. Nevertheless, some of the studies done with the truncated normalized Hamiltonians (we give details later) have considered even the possibility of small values of G. But, then, the following question arises: which of the equilibria belonging, for instance, to the Brouwer's model may be connected to the higher order averaged problem?, and to the full problem? When, (looking for relative equilibria) we only see G as a root of a polynomial equation, and we open the possibility for small values, we are dragged into the problem of the region of validity of the normalized Hamiltonian and expressions related to it. The issue of the validity of the normalized Hamiltonian has always been in the literature. As an example, the following paragraph, borrowed from Hori and Kozai (1975), put the situation this way: "The case of a large e and a small a that is the case of a small angular momentum G *seems* to offer another difficulty. In fact if G is small, the motion of the satellite approaches rectilinear and the perturbations would increase because of a close approach to the earth (if the earth be an oblate point mass). In view of a finite size of the earth, this case is realized by the motion of a missile rather than a satellite. This is a case where the averaging principle is not applied."

But having declared that, they really could not go further, trying to answer the question, without closed form expressions. We think one of the most valuable aspects of the closed form normalization of Coffey and Deprit (3) is that it goes directly to the core of the problem. Each term of the series expansion is made of two parts. One which contains G in the denominator, and the other part which may be considered as a polynomial function of three variables $P(\eta, s, C)$, (where $C = \cos g$). In other words, the form itself of the normalized Hamiltonian gives a first rough estimate condition for convergence

$$J_2\alpha^2\mu^2 \ll G^4. \tag{5}$$

Let us just mention that Healy uses $\beta = 1/(1+\eta)$ when he presents the expressions of the normal form. It is easy to see in those expressions that they always appear as $(e^2\beta)^j$ and $e^2\beta = 1 - \eta$; they may be put as polynomials in η. At the sixth order contribution (see p. 108) there is one factor in the terms factorized by $\cos 2g$ of the form e^2/η which (if not a typographical error) is not giving the full expression a polynomial form, as in the previous orders. Whether it is an error or not, it does not change this note. Moreover, notice that the question of convergence raised in this note may be also considered taking into account others of the formal expansions, like the one giving the elimination of the parallax. The reason for having chosen the normalized Hamiltonian is because it relates to the literature on the subject, and also because we have identified values which are relevant for the conclusion. Probably the same can be obtained using other expansions.

3 Relative equilibria: putting things in the astrodynamics context

The main problem or any model for satellite theory differs from the classic two-body system by its very nature. Meanwhile in the two-body model both bodies are taken as point masses, the satellite problem requires to keep in mind that one of the bodies cannot be taken as a point. Although the considerations of this note focus on satellites around planet-like bodies, we think some of them may be extended to a body with a more general type of figure.

Why the conditions for impacting or grazing orbits are not considered usually as the lower bound in satellite dynamics in Celestial Mechanics is an issue not easy to explain for us. It is even rather surprising if we consider that the topic of ballistic arcs and the possibility of using perturbation theories in its study was already discussed in the 70s (see for instance

Peters 1970; Hori 1973). A possible explanation might be that mathematics oriented papers in orbital dynamics, using techniques coming from the classic n-body problem, have underestimated this aspect; their only concern has always been the possibility of collisions (when at least one of the relative distances becomes zero), a question fully studied in that problem.

Apart, of course, from the beginning and end of a mission, in the astrodynamics community lower bounds (grazing orbits) is the natural ingredient for any mission design and control analysis. With that severe constraint taken into account, the issue of convergence is no longer a problem. Nevertheless the constraint defining the normalized Hamiltonian puts limits to integrals of the problem; an aspect, as we said in the Introduction, which has passed unnoticed.

3.1 Zero order approximation. A lower bound for L

As we know from any textbook on astrodynamics, the *zero order* of the satellite theory deals just with the physical constraint introduced by α. By zero order we mean to look first at our oblate planet as a sphere ($J_2 = 0$) of radius $\alpha > 0$ with homogeneous distribution of mass. Three types of orbits are associated with the satellite problem: non-impacting, grazing, and impacting orbits, these last ones including rectilinear. (We will not use the word "collision" which is usually reserved for conditions such that the particle reach the origin, when we assume point masses).

Non-impacting orbits. They are defined by the relation between the semi-major axis a, the eccentricity e and the equatorial radius:

$$a(1-e) > \alpha.$$

From this we get $a > \alpha,\ \ 0 \le e < 1 - \alpha/a$. In Delaunay variables: $L^2 = \mu a$ and $G = \sqrt{\mu a(1-e^2)}$, the conditions are

$$\mu\alpha < L^2 \qquad \text{and} \qquad \sqrt{\mu\alpha\left(2 - \frac{\mu\alpha}{L^2}\right)} < G \le L. \tag{6}$$

Grazing orbits. Those orbits are characterized by

$$a(1-e) = \alpha.$$

From this we get $a \ge \alpha,\ \ 0 \le e = 1 - \alpha/a < 1$. In Delaunay notation

$$\mu\alpha \le \tilde{L}^2, \qquad \text{and} \qquad \tilde{G} = \sqrt{\mu\alpha\left(2 - \frac{\mu\alpha}{\tilde{L}^2}\right)}. \tag{7}$$

When we deal with the two-body problem ($\alpha = 0$) we have $\tilde{G} = 0$.

As an orbit is defined by the three integrals (L, G, H), and considering the value of G for grazing orbits given by Eq. 7, we propose to introduce the value

$$H_0 = \pm\sqrt{\mu\alpha\left(2 - \frac{\mu\alpha}{L^2}\right)}.$$

Then, in contrast with the Kepler problem, the bounded motions around a sphere suggest to consider the domain of the norm of the angular momentum divided in two regions: (i) $G \in (|H_0|, L]$ where we guarantee no impacting conditions and, (ii) $G \in [|H^*|, L]$, where $0 \le |H^*| < |H_0|$. In this case we have impacting orbits when $|H^*| \le G < |H_0|$.

3.2 First order approximation: frozen-grazing orbits

Now we transfer the previous relations (6) and (7) to the averaged main problem of the artificial satellite theory and its *relative equilibria*. In other words, the inequalities above should be taken as the zero order contribution of relations to be expanded in powers of J_2. As we see below first order gives some basic expressions.

The first order truncated normalized Hamiltonian is

$$\mathcal{K}^{(1)} = -\frac{\mu^2}{2L^2} + J_2\frac{\alpha^2\mu^4}{G^3L^3}\frac{1}{4}\left(1 - 3\frac{H^2}{G^2}\right), \tag{8}$$

and the corresponding reduced system

$$\dot{G} = 0, \quad \dot{g} = -J_2\frac{3}{4}\frac{\alpha^2\mu^4}{L^3}\frac{1}{G^4}\left(1 - 5\frac{H^2}{G^2}\right). \tag{9}$$

Thus, at this approximation G is an integral of this system. Note that the range of values of G satisfies

$$|\mathcal{K} - \mathcal{K}^{(1)}| \approx \mathcal{O}(J_2^2),$$

in agreement with we have already said (see (5)).

Relative solutions of this system give *elliptic orbits with stationary perigee at critical inclination* $c^2 = 1/5$. We have

$$G_0^2 = 5H^2, \quad 0 \ll |H| \leq \frac{L}{\sqrt{5}}. \tag{10}$$

Apart from the lower bound for H which is a consequence of the lower bound for G, the upper bound seems to have been introduced by Cushman, referring it to Orlov. Then, if we consider the mean motions of the system

$$\dot{\ell} = \frac{\mu^2}{L^3} + J_2\frac{3}{4}\frac{\alpha^2\mu^4}{G^3L^4}\left(1 - 3\frac{H^2}{G^2}\right), \quad \dot{h} = -J_2\frac{3}{2}\frac{\alpha^2\mu^4 H}{G^5L^3}, \tag{11}$$

and replace the value $G_0^2 = 5H^2$ in Eq. 11 we obtain the following *frozen mean motions*

$$n_\ell^{(1)} = \frac{\mu^2}{L^3} + J_2\alpha^2\mu^4\frac{3}{25\sqrt{5}}\frac{1}{L^4|H|^3}, \quad n_h^{(1)} = -J_2\alpha^2\mu^4\frac{3}{50\sqrt{5}}\frac{1}{L^3H^4}.$$

These expressions show immediately that if (forgetting about the lower bound for H) we take small values for H, they have no meaning: the satellite in the averaged orbit will move with a very different mean motion and the averaged ellipses will be very fast rotating. But this 'pathological' situation is clarified when we realized that for values of H of the order of J_2 or smaller, the treatment of the problem and its normalization should be different (see Sect. 4.2).

Then, considering astrodynamics applications, if we impose to $G_0 = \sqrt{5}|H|$ the constraint (7) given above, in order to have non-impacting relative equilibria the integrals L and H have to satisfy the conditions:

$$H \in [H_{0,1}, H_{0,0}] = \left[\sqrt{\frac{\alpha\mu}{5}\left(2 - \frac{\alpha\mu}{L^2}\right)}, \quad \frac{L}{\sqrt{5}}\right] \quad \text{and} \quad \mu\alpha < L^2, \tag{12}$$

and the value of the eccentricity of the grazing orbits is

$$e = 1 - \frac{\alpha\mu}{L^2}.$$

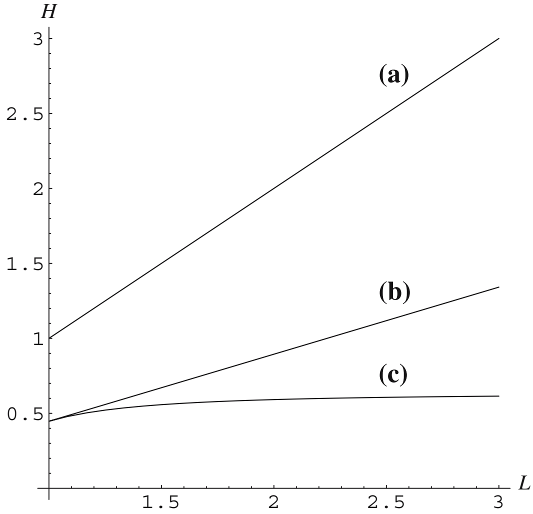

Fig. 1 Relevant lines in the (L, H) plane: (**a**) the border defined by $H = L$ (circular equatorial); (**b**) bifurcation line $H = L/\sqrt{5}$ of circular orbits at critical inclination, and; (**c**) line of grazing frozen orbits $H = \sqrt{(2 - 1/L^2)/5}$, (we have taken $\mu = 1, \alpha = 1$, and we only present the case $H > 0$)

In Fig. 1 we show the two relevant lines defining the interval given by (12) as function of L, joint with the line $H = L$ which gives the border of the parameter domain.

3.3 Second order refinements

The first order analysis, based on $\mathcal{K}^{(1)}$, has given us a 'critical inclination' where all the ellipses have fixed perigee. Thus, at this order there is infinite number of relative equilibria; in other words, the system is 'degenerate' and the analysis has to be pushed to the following order dealing with $\mathcal{K}^{(2)}$, trying to understand what really happens. As we said in Sect. 1, this is what was done in the 80s by Coffey et al. They found two pitchfork bifurcations stemming from circular orbits (Cushman 1988, gave later a more refined analysis). The corresponding expressions giving the four relative equilibria, defining the corresponding 'elliptic frozen orbits,' two stable and two unstable, are captured by the expressions Eqs. 24_1 and 25_1 of CDM, (or the similar ones given by Cushman).

Far from circular orbits the reduced system may be studied with Delaunay variables. The system defined by the Brouwer model is (see Eqs. 23_1 and 23_2 of CDM)

$$\dot{g} = \frac{\partial \mathcal{K}^{(2)}}{\partial G} = 0, \quad \dot{G} = -\frac{\partial \mathcal{K}^{(2)}}{\partial g} = 0. \tag{13}$$

Looking for equilibria it is easy to check that, apart from the poles of the S^2-sphere which always exist where the system is defined, all the other possible roots will be found when $g = k\pi/2$. From the two pairs $g = 0, \pi$ and $g = \pi/2, 3\pi/2$ of the first equation, if we replace in the second equation (13), we obtain two equations

$$N_1 = 32(4 - 5s^2) + J_2 \frac{\alpha^2 \mu^2}{G^4} \left[\gamma_{0,0} + \gamma_{2,0} + \gamma_{0,1}\eta + (\gamma_{0,2} + \gamma_{2,2})\eta^2\right] = 0$$

and

$$N_2 = 32(4 - 5s^2) + J_2 \frac{\alpha^2 \mu^2}{G^4} \left[\gamma_{0,0} - \gamma_{2,0} + \gamma_{0,1}\eta + (\gamma_{0,2} - \gamma_{2,2})\eta^2\right] = 0,$$

respectively, where $\gamma_{i,j}$ are given in CDM. We briefly reproduce the study done by CDM for these equations, but in a slightly different form.

Let us consider $N_2 = 0$ which corresponds to the unstable frozen orbits (the other equation can be treated the same way). Replacing $s^2 = 1 - H^2/G^2$, $\eta = G/L$ and $J_2^* = \alpha^2\mu^2 J_2$ we obtain a polynomial equation $\mathcal{P}_8(G; L, H, J_2^*) = 0$ of degree 8 in G. Putting aside the convergence constraint, we are interested in discussing the possible existence of other roots $|H| \le G_i \le L$. We see that this equation is biquadratic in H. Thus, we obtain immediately an expression $H = H(G; L, J_2^*)$ which shows that there is always one real root G_2 satisfying the dynamical constraint $H < G_2 < L$, and that this root goes to zero when H goes to zero.

Instead of $G_2 = G(L, H; J_2)$ at critical inclination, we think it is more illustrative of the characteristic of the equilibria, to see $N_2 = 0$ as a relation between the eccentricity and the inclination of the frozen orbits $N_2 = N_2(c, e; L, J_2) = 0$. Denoting

$$J = \frac{\alpha^2\mu^2}{L^4} J_2,$$

we have

$$c^2 = \frac{(-189 + 96\eta + 175\eta^2)J - 80\eta^4 + 80\sqrt{\eta^8 + AJ + BJ^2}}{5(11 + 72\eta + 63\eta^2)J}, \tag{14}$$

with

$$A = -\frac{1}{5}(-25 + 3\eta + 14\eta^2)\eta^4,$$
$$B = \frac{1}{400}(2401 - 1248\eta - 3254\eta^2 + 840\eta^3 + 1225\eta^4).$$

Although this function (14) is well defined for the whole range $\eta \in [0, 1]$, it only has sense in the range of convergence. At least for the non-impacting and grazing domain we may simplify Eq. 14 computing the series expansion

$$c = \frac{1}{\sqrt{5}}\left(1 + \frac{-9 + 7\eta^2}{20\eta^4}\frac{\alpha^2\mu^2}{L^4} J_2\right) + \mathcal{O}(J_2^2). \tag{15}$$

In particular we may obtain the inclination of the frozen-grazing orbit replacing η for the value given in Eq. 7

$$\tilde{\eta} = \sqrt{\frac{\mu\alpha}{\tilde{L}^2}\left(2 - \frac{\mu\alpha}{\tilde{L}^2}\right)},$$

although we do not print the expression. For instance, taking units $\mu = \alpha = 1$, for $L = 2$ we have $\tilde{\eta} = 0.661438$. Moreover, note that the expression given by CDM can be obtained readily from Eq. 15.

4 No frozen orbits for small values of *H*

When the third component of the angular momentum H is small, the previous analysis leaves the evolution of elliptic frozen orbits as an open question, because convergence of the expressions used has to be taken into account. In this section we present two approaches in order to answer this question. First we consider the Coffey–Deprit model, trying to refine the domain of convergence. We find again a lower bound for the normalization, but not refinement of the one given in Sect. 2. The second approach shows that, from the structure of the Hamiltonian, we may conclude that there is not elliptic frozen orbits when H is small.

4.1 'New equilibria' but not convergence

In Sect. 2, we have identified a rough lower bound for convergence. May that bound be refined dealing with the relative equilibria of the Coffey–Deprit model? The finding of 'new equilibria' will really mean another constraint for G, because no more roots can be added to the second order analysis.

We consider the third order truncation (Coffey–Deprit model) of the normalized Hamiltonian. Are there other equilibria when third order terms are included? It is not difficult to find a segment of values (H_l, H_u) of the integral H, functions of L and J_2, where we identify eight relative equilibria in the corresponding differential system. The explanation is that the well known *elliptic frozen orbits* (stable and unstable relative equilibria in the twice reduced space) of the second order truncation (Brouwer model), undergo two pitchfork bifurcations, if they are approached within the third order approximation. As the reader will see, we only show numerical evidence of the evolution of the number of equilibria and their stability character.

The reduced system defined by the Coffey–Deprit Hamiltonian function Eq. 3 is

$$\dot{g} = \frac{\partial \tilde{\mathcal{K}}}{\partial G}, \quad \dot{G} = -\frac{\partial \tilde{\mathcal{K}}}{\partial g}, \tag{16}$$

where, dropping the Kepler part and rescaling in Eq. 3, we have

$$\tilde{\mathcal{K}} = \mathcal{K}_1 + \frac{1}{2!} J_2 \mathcal{K}_2 + \frac{1}{3!} J_2^2 \mathcal{K}_3.$$

Then, the equations satisfied by the equilibria may be written as

$$\frac{\partial \tilde{\mathcal{K}}}{\partial g} = P_1(G; L, H, J_2) \sin 2g = 0, \tag{17}$$

$$\frac{\partial \tilde{\mathcal{K}}}{\partial G} = P_2(G; L, H, J_2) + P_3(G; L, H, J_2) \cos 2g = 0. \tag{18}$$

Starting the search of frozen orbits at the north pole of the twice reduced space (using CDM coordinates), as expected, there are frozen elliptic equilibria of Eq. 16 which evolve from two pitchfork bifurcations of circular orbits as in the second order. The third order terms only bring slight refinements to the expressions of the bifurcations lines already known. Moreover, the south pole of the reduced space is also an equilibrium, meanwhile the normalized Hamiltonian is convergent.

Far from circular orbits the reduced system may be studied with Delaunay variables. What we are after is the possible existence of other roots when we are far from the neighborhood of circular orbits. From the previous system, searching for other equilibria out of the principal meridians $\sin 2g = 0$, first we ought to have

$$P_1(G; L, H, J_2) = 0.$$

Solving this equation and assuming $\tilde{G} = \tilde{G}(L, H; J_2)$ is one of the roots, if we replace in Eq. 18 we will get

$$\cos 2g = -\frac{P_2(\tilde{G}, L, H; J_2)}{P_3(\tilde{G}, L, H; J_2)}.$$

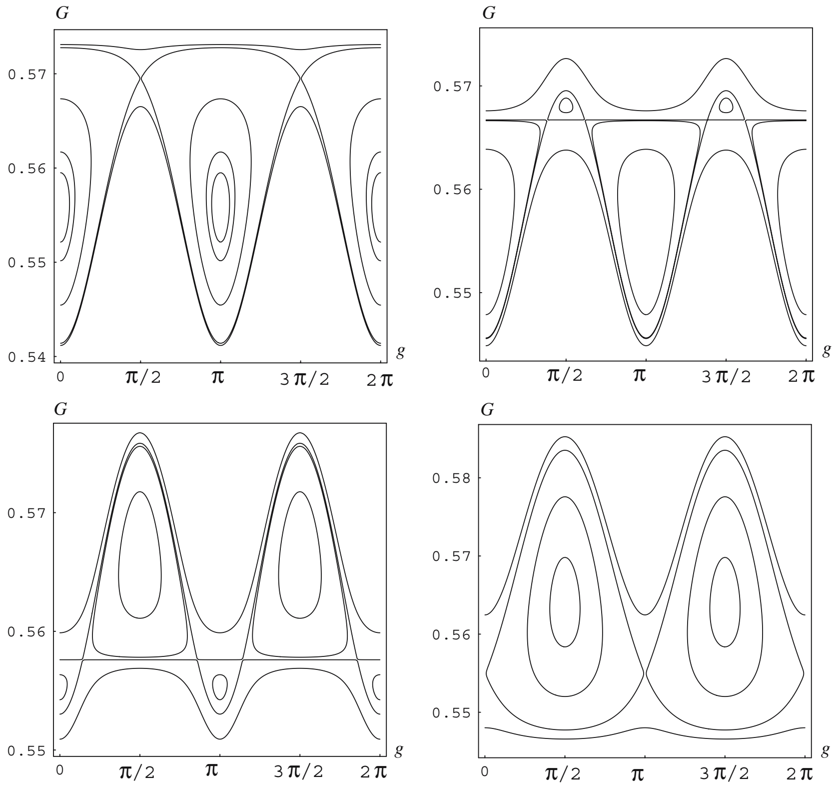

Fig. 2 A sequence of the switch of stable–unstable equilibria at the 'elliptic frozen orbits' zone when the third order terms are taken into account. It is the result of two pitchfork bifurcations of elliptic orbits when H enters and leaves the interval $(H_l,\ H_u)$ from above. Stemming from unstable orbits at the meridian $g = \pi/2, 3\pi/2$ they migrate towards the stable equilibria at the meridian $g = 0, \pi$ which become unstable. In these figures we have fixed $L = 2$ and $J_2 = 0.01$

What remains is to identify the range of values (L, H) in the plane of dynamical parameters, such that

$$-1 \leq -\frac{P_2(\tilde{G}, L, H)}{P_3(\tilde{G}, L, H)} \leq 1.$$

In fact, what we need to obtain are the limit lines $H_l = H_l(L; J_2)$ and $H_u = H_u(L; J_2)$ corresponding to -1 and 1, respectively. The result is presented in Figs. 2 and 3 which gather our analysis. As neither the expression for $\tilde{G}$ nor the functions $H_l = H_l(L; J_2)$ or $H_u = H_u(L; J_2)$ can be obtained explicitly, we do not show the numerical computations carried out.

The explanation of what happens is as follows. We find a segment of values (H_l, H_u) of the integral H, functions of L and J_2, where we identify eight relative equilibria in the corresponding differential system. This means that the well known *elliptic frozen orbits* (stable and unstable relative equilibria in the twice reduced space) of the second order truncation (Brouwer model), undergo pitchfork bifurcations, if they are approached within the third order approximation. Let us fix values for L and J_2 and consider a value of $\overline{H}$ after the two pitchfork bifurcations of circular orbits have taken place. Then, for a range of values

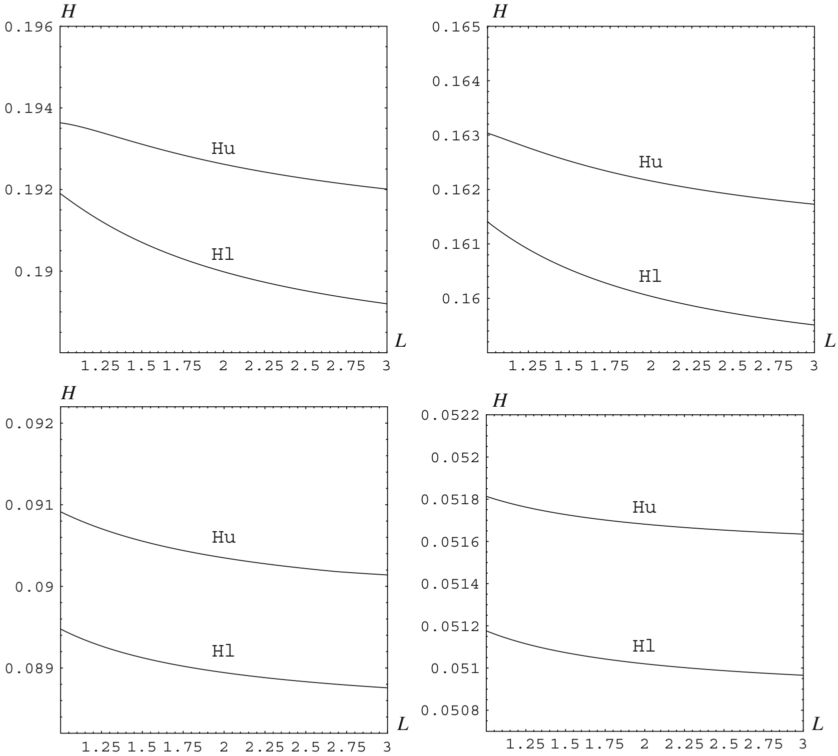

Fig. 3 The relative influence of J_2. As a zoom in the plane of dynamical parameters (L, H), we show bifurcations lines $H_u = H_u(L; J_2)$ and $H_l = H_l(L; J_2)$ when the third order influence is taken into account. They correspond to the following values of J_2: 0.2, 0.1, 0.01, 0.001, respectively. Note that in all of them we restrict to the interval $L \in [1, 3]$

$H < \overline{H}$ we have still a 'critical inclination zone' with four equilibria: two stable $E_{1,2}$ for $g = 0, \pi$, and two unstable $E_{3,4}$ for $g = \pi/2, 3\pi/2$, already known from the second order analysis. Then, when $H = H_u < \overline{H}$ there is a pitchfork bifurcation of the equilibria $E_{3,4}$ which become stable. For $H \in (H_l, H_u)$ the 'critial inclination zone' is different: due to the third order terms, we have the previous equilibria $E_{1,2,3,4}$ which are stable and four new equilibria $E_{5,6,7,8}$ which are unstable. Then, there is a value $H = H_l$ such that the equilibria $E_{5,6,7,8}$ reach $E_{1,2}$ through a new pitchfork bifurcation, and for $H < H_l$ the equilibria $E_{1,2}$ become unstable; the other two remaining $E_{3,4}$ continue to be stable. In short, we may say that in this range of H there has occurred a 'rotation of stability,' which we present in Fig. 2. Complementing the previous figure, in Fig. 3 we present a sequence which is a zoom in the *plane of dynamical parameters* (L, H), where we show the two new bifurcation lines when the third order influence is taken into account; they correpond to several values of J_2: 0.2, 0.1, 0.01 and 0.001, respectively. Note that in all of them we restrict ourselves to the interval $L \in [1, 3]$.

Are there other equilibria in the Coffey–Deprit model? For the purpose of this note, we do not need to answer that question. The existence of the equilibria $E_{5,6,7,8}$ is sufficient. According to perturbation theory, after the degeneracy of the first order is corrected with the second order contribution, bringing the number of relative equilibria to a maximum of six, no more roots are expected if higher order terms of the normalization are included; only changes of their values of the order of the small parameter. Thus, the existence of the previous interval where new roots appear only means that we are out of the domain of convergence of the normalized Hamiltonian. Those equilibria, proper of the Coffey–Deprit model, have nothing to do with the main problem. Indeed, as a quick computation shows, when we look for values of $\tilde{G}$ related to Fig. 2, we find that $\tilde{G}^4 \approx J_2$, which contradicts condition (5). Although, this does not infer an improvement of the lower bound. In other words, the values of $\tilde{G}$ related to the previous interval, help to confirm the lower bound, intrinsic of the normalization, previously identify. We conjecture that the inclusion of higher order terms will bring new restrictions to the lower bound.

4.2 On the almost polar orbits

In order to put lower bounds for some formulas related to frozen orbits (apart from the mentioned grazing condition) do we really need to invoke lack of convergence in the expressions used? As the proper expressions suggest, the question to answer is simply: what happens when the integral $|H| \in [0, L]$ takes very small values, say $|H| \leq \sqrt{\alpha\mu J_2}$?

In our understanding this tells that, again, we should look at the Hamiltonian function (2) of the main problem, but this time written explicitly

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} + J_2\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2\left[\frac{1}{4}(1 - 3\cos 2\theta) + \frac{3}{4}\frac{N^2}{\Theta^2}(-1 + \cos 2\theta)\right]. \quad (19)$$

Apart from the quadrature corresponding to the precessional motion of the node of the orbital plane $\nu = \int(\partial\mathcal{H}/\partial N)\,dt$, we see that the function (19) defines a two-degrees of freedom system $\mathcal{H}(r, \theta, R, \Theta)$ with respect to the orbital plane. The relevant point for us is to see the Hamiltonian function as an uni-parametric family of Hamiltonians defined by $\mathcal{H}_N \equiv \mathcal{H}(r, \theta, R, \Theta; J_2, N)$, i.e. where N (remember N is the name of the integral H in polar-nodal variables) is taken as a parameter rather than as an integral. In other words, denoting σ the dimensionless quantity

$$\sigma = J_2\frac{N^2}{\alpha\mu},$$

the main problem may be written as

$$\begin{aligned}\mathcal{H}_\sigma = {} & \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} + J_2\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2\frac{1}{4}(1 - 3\cos 2\theta)\\ & + \sigma\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2\frac{\alpha\mu}{\Theta^2}\frac{3}{4}(-1 + \cos 2\theta),\end{aligned} \quad (20)$$

with $N < \Theta$. When we consider $N \in (0, \sqrt{J_2\alpha\mu}\,]$, then σ is similar or even smaller than J_2^2. Then, after a Delaunay transformation and the corresponding normalization, it is easy to see that on the $S^2(L, H)$ reduced orbital space the relative equilibria coming from (20) reduce to the north pole, i.e., to a circular stable orbit at inclination $\cos I = H/L$ and the flow as a rotation; moreover, the neighborhood of the south pole cannot be reached because, as H is small, there is no convergence.

These comments seem to be extensible to the 'frozen orbits' of the zonal problem, where other harmonic coefficients are considered (see Coffey et al. 1994), in particular for families of almost polar orbits. In that case the normalization procedure may be simplified, but we will not touch that point in this note (Ferrer et al. 2007). Comparison with the study done by Lara (1997) in the polar case is also in progress.

5 Conclusions

Coffey et al. and Cushman left unfinished their study of the critical inclination in satellite theory: no analysis was made of the evolution of the elliptic frozen orbits emanating from circular orbits when one of the integrals, the third component of the angular momentum, takes small values. In this note we approach this issue and consider three related aspects associated to relative equilibria analysis. The first aspect refers to the role of the physical parameters; we point out that they define a lower bound for the domain of convergence of the normalized Hamiltonian.

Moreover, we give expressions, function of the equatorial radius of the main body, in order to identify conditions for non-collision frozen orbits. This issue was already in the literature (see for instance Hough 1981a, b, well before the work of Cushman and Deprit et al.), although no mention was made at the time.

The third aspect studied is the role played by the integral H. When it is rather small, say of the order of $\sqrt{J_2}$ or smaller, we show that there are not frozen orbits. Finally, we point out that most of the contents of this note applies also when more realistic models are taken into account, like the ones including other zonal or tesseral harmonics.

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