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A stability version of Hölder's inequality

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Abstract

We present a stability version of Hölder's inequality, incorporating an extra term that measures the deviation from equality. Applications are given.

Keywords: Hölder's inequality

1. Introduction

In the field of geometric inequalities, the expression *Bonnesen type* is used after Bonnesen classical refinement of the isoperimetric inequality (cf., for instance, [15,16]), where the deviation from the case of equality (the disk) is given in terms of the outer radius and the inradius of a bounded convex body. The term *stability type* inequality is also used in a related way (cf. [9]), meaning that if the deviation from equality is "small," then the objects under consideration must be "close" to the extremal object.

Here we explore the question of what a Bonnesen or stability version of Hölder's inequality should look like, as we move away from the equality case. Since the functions f and g involved in Hölder's inequality will usually belong to different spaces, before they can be compared we need to map these functions, with controlled distortion, into a "common measuring ground." The way we choose to do this is by first normalizing, and then applying the Mazur map from L^p and L^q to L^2. For nonnegative functions in the unit sphere of L^p the Mazur map into L^2 is defined by $f \mapsto f^{p/2}$. We will be able to utilize its well-known properties (cf., for instance, [4]) to obtain useful estimates.

As a model for the stability version of Hölder's inequality, we use the (real) Hilbert space parallelogram identity, suitably rearranged under the assumption that the vectors are nonzero (see (2.0.2) below). With (2.0.2) in mind we obtain a natural, straightforward generalization of the parallelogram identity, valid for $1 < p < \infty$, though when $p \neq 2$ equality will of course be lost, cf. (2.2.1). After one has decided which inequality to prove, the argument is standard. In fact, it is *the* standard argument: From a refined Young's inequality one obtains a refined Hölder inequality, which

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in turn entails a refined triangle inequality, which (together with a simple additional observation) yields the uniform convexity of L^p spaces in the real valued case, with optimal power type estimates for the modulus of convexity.

Like the parallelogram identity in the Hilbert space setting, (2.2.1) brings to the fore the geometry of L^p spaces, and conveys essentially the same information: In order for $\|fg\|_1$ to be close to $\|f\|_p\|g\|_q$, the angle between the nonnegative L^2 functions $|f|^{p/2}$ and $|g|^{q/2}$ must be small, with equality in $\|fg\|_1 \leqslant \|f\|_p\|g\|_q$ precisely when the angle is zero. Since Hölder's inequality is one of the most often used inequalities, the refinement given here is likely to have repercussions far beyond the few applications presented below.

The paper is organized as follows. Section 2 contains the basic inequality and its proof, together with the precedents I have been able to find, and a small discussion as to why some plausible improvements of (2.2.1) cannot hold. Section 3 establishes a few direct consequences regarding bounds on interpolated norms. Specializing the previous remark about angles to the function 1 on a probability space, we obtain a stability version of the following standard application of Hölder's inequality: If $0 < r < s$, then every $f \in L^s$ satisfies $\|f\|_r \leqslant \|f\|_s$, with equality if and only if $|f|$ is constant. As we noted, the norms $\|f\|_s$ and $\|f\|_r$ will be close if and only if the angle between 1 and $|f|^{s/2}$ is small (cf. Theorem 3.1). Expressing this result in terms of the variance of $|f|^{s/2}$, we shall see that $\|f\|_s$ and $\|f\|_r$ are close if and only if the normalized variance $\operatorname{Var}(|f|^{s/2}/\||f|^{s/2}\|_2)$ is sufficiently small, cf. Corollary 3.2. These results provide qualitative information about the behavior of L^p norms, which apparently had not been noticed before. Finally, Section 4 contains a sharpened triangle inequality, leading to the proof of uniform convexity announced above.

We work on an arbitrary measure space $(X, \mathcal{A}, \mu)$, whose mention will usually be omitted; to avoid trivialities we assume that μ is not identically zero, and (when dealing with uniform convexity) that X contains at least two points.

2. The basic inequality

In this paper p and q always denote conjugate exponents, i.e., $q = p/(p-1)$, and unless otherwise stated, it is understood that $f \in L^p$, $g \in L^q$ and neither function is zero almost everywhere. To motivate the variant of Hölder's inequality given below, let us consider first the situation in a real Hilbert space setting. From the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2.0.1}$$

we get, after expanding $\|x+y\|^2$, replacing x by tx, taking $t = \|y\|/\|x\|$, and factoring $\|x\|\|y\|$, the equality

$$(x, y) = \|x\|\|y\|\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right) \tag{2.0.2}$$

valid for nonzero x and y. We follow this line of thought in the L^p setting, using (2.0.2) as a model. Observe that the identity (2.0.2) can be regarded as a stability version (and also a proof) of the Cauchy–Schwarz inequality.

The first step is to refine Young's inequality $u^p/p + v^q/q - uv \geqslant 0$.

Lemma 2.1. *Let $1 < p \leqslant 2$ and let q be its conjugate exponent. Then for all $u, v \geqslant 0$,*

$$\frac{1}{q}\left(u^{p/2} - v^{q/2}\right)^2 \leqslant \frac{u^p}{p} + \frac{v^q}{q} - uv \leqslant \frac{1}{p}\left(u^{p/2} - v^{q/2}\right)^2. \tag{2.1.1}$$

Proof. If $p = 2 = q$ the result is trivial, so assume $1 < p < 2$. We prove the first inequality; the second can be obtained via an essentially identical argument, by interchanging the roles of p and q, and of u and v. If either $u = 0$ or $v = 0$, formula (2.1.1) is obviously true. Fix p, fix $u > 0$, and suppose $v > 0$. Expanding the square and simplifying, we see that it is enough to check the following inequality:

$$f(v) := \frac{2-p}{p}u^p + \frac{2}{q}u^{p/2}v^{q/2} - uv \geqslant 0. \tag{2.1.2}$$

Now $v = u^{p-1}$ is the unique solution of $f'(v) = 0$. Since $f'' > 0$, $f(u^{p-1}) = 0$ is the global minimum of f. □

An extension of (2.0.2) to the case $1 < p < \infty$ follows now by repeating the steps in the usual derivation of Hölder's inequality from Young's inequality. Only minimal modifications to the Hilbert space argument given above are needed, though of course, the equality becomes a two sided inequality when $p \neq 2$. We write $t_+ := \max\{t, 0\}$ for the positive part of a real number or a real valued function, and $t_+^r := (\max\{t, 0\})^r$, so the maximum is taken first.

The left-hand side of the identity $\|f\|_p^{p/2} = \||f|^{p/2}\|_2$ seems to be typographically more convenient and easier to read than the right-hand side, so we will use it below. However, it makes it less obvious that in (2.2.1) the functions $|f|^{p/2}/\|f\|_p^{p/2}$ and $|g|^{q/2}/\|g\|_q^{q/2}$ are simply norm 1 vectors in L^2 (so we are in fact dealing with the angle between $|f|^{p/2}$ and $|g|^{q/2}$, cf. Remark 2.3).

Theorem 2.2. *Let $1 < p < \infty$ and let $q = p/(p-1)$ be its conjugate exponent. If $f \in L^p$, $g \in L^q$, $\|f\|_p, \|g\|_q > 0$, and $1 < p \leqslant 2$, then*

$$\|f\|_p\|g\|_q\left(1 - \frac{1}{p}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}} - \frac{|g|^{q/2}}{\|g\|_q^{q/2}}\right\|_2^2\right)_+ \leqslant \|fg\|_1 \leqslant \|f\|_p\|g\|_q\left(1 - \frac{1}{q}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}} - \frac{|g|^{q/2}}{\|g\|_q^{q/2}}\right\|_2^2\right), \tag{2.2.1}$$

while if $2 \leqslant p < \infty$, the terms $1/p$ and $1/q$ exchange their positions in the preceding inequalities.

Proof. Suppose $1 < p \leqslant 2$. Write $u = |f(x)|$ and $v = |g(x)|$ in (2.1.1), integrate, substitute tf for f, and set $t = \|g\|_q^{1/(p-1)}/\|f\|_p$. Now (2.2.1) immediately follows. If $2 \leqslant p < \infty$, just interchange the roles of p and q. □

Of course, when $p = 2$ the inequality (2.2.1) follows from (2.0.2), and in fact, it is identical to it, save for the fact that only nonnegative functions appear in (2.2.1).

The reason why we take the positive part in the left-hand side of (2.2.1), is that in some inequalities given below we will need to take powers of the corresponding quantities.

Remark 2.3. Recall that in a real inner product space, the angle $\angle(x, y)$ between x and y is defined by

$$\angle(x, y) := \arccos\left(\frac{(x, y)}{\|x\|\|y\|}\right) = \arccos\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right),$$

where the second equality follows from (2.0.2). Actually, the simpler expression $\theta(x, y) := \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|$, giving the length of the segment between $x/\|x\|$ and $y/\|y\|$, is often taken as the *definition* of angle in a general Banach space (cf., for instance, [5, p. 403]). In the real Hilbert space setting, $\angle(x, y)$ and $\theta(x, y)$ are clearly comparable quantities (in fact, $\theta(x, y) \leqslant \angle(x, y) \leqslant (\pi/2)\theta(x, y)$) so up to a constant it does not matter which one is used. Thus, the geometric content of (2.2.1) is clear: $\|fg\|_1 \approx \|f\|_p\|g\|_q$ if and only if the angle $\angle(|f|^{p/2}, |g|^{q/2})$ is small. Note also that the same term $\theta^2(|f|^{p/2}, |g|^{q/2})$ appears both on the left and on the right-hand sides of (2.2.1); hence, the exponent 2 cannot be improved. This helps to explain why from (2.2.1) we obtain optimal asymptotic power type estimates for the modulus of convexity of $L^p(X, \mathbb{R})$ spaces.

Observe that if f and g have disjoint supports then (2.2.1) becomes

$$\|f\|_p\|g\|_q\left(1 - \frac{2}{p}\right)_+ \leqslant \|fg\|_1 = 0 \leqslant \|f\|_p\|g\|_q\left(1 - \frac{2}{q}\right). \tag{2.3.1}$$

Hence, the right-hand side bound worsens as $p \to 1$ (and $q \to \infty$). Note also that the constant $1/2$ appears, instead of $1/p$ and $1/q$, both in (2.0.2) above and in (2.4.4) below. Thus, it is natural to ask whether it is possible to improve at least one of the factors $1/p$, $1/q$ in (2.2.1), replacing it by $1/2$ (of course, when supports are disjoint we cannot do better than writing 0 on the left-hand side, but under less than full orthogonality, the change from $1/p$ to $1/2$ might be useful). Next we show that such change is not possible.

Example 2.4. Let $1 < p < 2$. Replacing $1/q$ by $1/2$ in the right-hand side of (2.2.1) and simplifying we find that this modification of the second inequality is equivalent to

$$\int |fg| \leqslant \|f\|_p^{1-p/2}\|g\|_q^{1-q/2}\int |f|^{p/2}|g|^{q/2}. \tag{2.4.1}$$

Likewise, replacing $1/p$ by $1/2$ in the left-hand side of (2.2.1) leads to

$$\int |fg| \geqslant \|f\|_p^{1-p/2}\|g\|_q^{1-q/2}\int |f|^{p/2}|g|^{q/2}. \tag{2.4.2}$$

It is easy to find examples showing that neither (2.4.1) nor (2.4.2) hold. Take, for instance, $f \equiv 1$ on $[0, 1]$ and $g = 2\chi_{[0,1/2]}$. Then $\|g\|_q = 2^{1-1/q}$, so $1 = \int fg < \|g\|_q^{1-q/2} \int |g|^{q/2} = 2^{1/2-1/q}$ and thus (2.4.2) fails. Choosing now $g \equiv 1$ and $f = 2\chi_{[0,1/2]}$ we have $\|f\|_p = 2^{1-1/p}$ and $1 > \|f\|_p^{1-p/2} \int |f|^{p/2} = 2^{1/2-1/p}$, so (2.4.1) does not hold either.

A more indirect argument shows that in fact $1/q$ cannot be replaced by any fixed constant $c \in (0, 1/2)$ (independent of p, or equivalently, of q). Since (2.2.1) can be used to prove the uniform convexity of L^p for $p > 1$, if there were such a c, then the upper bound in (2.2.1) would not degenerate as $p \downarrow 1$, and we would be able to show that the modulus of convexity of L^p is independent of p for every $p \in (1, 2]$, an obviously false result.

Despite its obvious interest, not much work has been done, as far as I know, regarding stability versions of Hölder's inequality. I am aware of two previous articles giving bounds for the deviation from the case of equality. In [7] the following result is presented:

$$0 \leqslant 1 - \frac{(|f|, |g|)}{\|f\|_p \|g\|_q} \leqslant \left(\frac{|f|^p}{\|f\|_p^p} - \frac{|g|^q}{\|g\|_q^q}, \frac{1}{q} \log|f| - \frac{1}{p} \log|g| \right) \leqslant \log\left[\frac{(|f|^{\frac{1+pq}{q}}, |g|^{-\frac{1}{p}})(|g|^{\frac{1+pq}{p}}, |f|^{-\frac{1}{q}})}{\|f\|_p^p \|g\|_q^q} \right], \tag{2.4.3}$$

where $(f, g) := \int fg$. Note that (2.4.3) does not coincide with the rearranged parallelogram identity (2.0.2) when $p = q = 2$.

An inequality more closely related to (2.2.1), which for nonnegative functions does extend (2.0.2), appears in [17]. The argument is actually the same as the one used here (and in the standard proof of Hölder's inequality), save for the fact that the initial refinement of Young's inequality is different from (2.1.1). Suppose $f, g \geqslant 0$. By Theorem 2 of [17], if $1 < q \leqslant 2 \leqslant p < \infty$, then

$$\frac{1}{2} \frac{\|g^{2-q}(f\|g\|_q^{q/p} - g^{q-1}\|f\|_p)^2\|_1}{\|f\|_p \|g\|_q^{q/p}} \leqslant \|f\|_p \|g\|_q - \|fg\|_1 \leqslant \frac{1}{2} \frac{\|f^{2-p}(g\|f\|_p^{p/q} - f^{p-1}\|g\|_q)^2\|_1}{\|f\|_p^{p/q} \|g\|_q}. \tag{2.4.4}$$

In addition to the factor $1/2$ mentioned before, there are other differences between (2.4.4) and (2.2.1). Note, for instance, that every term in (2.2.1) is finite, while for $p > 2$, whenever the support of g is not contained in the support of f the right-hand side of (2.4.4) blows up.

After submitting this paper I have come accross the article [8], where a refinement of Hölder's inequality is obtained by using the positive definiteness of the Gram matrix. Write $m := \min\{p^{-1}, q^{-1}\}$. Under the usual hypotheses, Theorem 2.3 of [8] states that

$$(f, g) \leqslant \|f\|_p \|g\|_q (1 - r)^m, \tag{2.4.5}$$

where r is an explicitly defined function of $f^{p/2}$, $g^{q/2}$ and a third normalized vector $h \in L^2$. Both inequalities (2.4.5) and (2.2.1) have in common the use of L^2 to bound the deviation from equality. As differences, we note that (2.4.5) is one sided, and it does not reduce to the rearranged parallelogram identity when $p = q = 2$.

Remark 2.5. It is easy to give a stability version of the following standard variant of Hölder's inequality: If $r > 0$, $p^{-1} + q^{-1} = r^{-1}$, $f \in L^p$, and $g \in L^q$, then $\|fg\|_r \leqslant \|f\|_p \|g\|_q$. From it and an induction argument, stability versions for multiple products can be obtained, that is, for the inequality $\|\prod_{i=1}^n f_i\|_r \leqslant \prod_{i=1}^n \|f_i\|_{p_i}$, where $f_i \in L^{p_i}$ and $\sum_{i=1}^n p_i^{-1} = r^{-1}$.

3. Interpolation-type consequences

In this section we derive some immediate interpolation-type results. Note that

$$\left(1 - \frac{1}{q} \left\| \frac{|f|^{p/2}}{\|f\|_p^{p/2}} - \frac{|g|^{q/2}}{\|g\|_q^{q/2}} \right\|_2^2 \right) = 1 - \frac{2}{q} \left(1 - \frac{\int |f|^{p/2} |g|^{q/2}}{(\int |f|^p)^{1/2} (\int |g|^q)^{1/2}} \right), \tag{3.0.1}$$

and these quantities are strictly positive when $q > 2$. In what follows, both expressions will be used.

Recall that on a probability space, if $0 < r < s$, then every $f \in L^s$ satisfies $\|f\|_r \leqslant \|f\|_s$, a fact that follows either from Jensen's inequality, or by writing $|f|$ as the product $|f| \cdot 1$ and then applying Hölder's inequality. From the equality case in either Jensen or Hölder inequalities, we have $\|f\|_r < \|f\|_s$ unless $|f|$ is constant. This suggests that the deviation of $|f|$ (or more precisely, of its normalized image under the Mazur map) from its mean value can be used to obtain finer bounds.

Theorem 3.1. *Let $0 < r < s < \infty$, and let $f \in L^s$ satisfy $\|f\|_s > 0$. If $s \leqslant 2r$, then*

$$\|f\|_s\left[1-\frac{2r}{s}\left(1-\frac{\||f|^{s/2}\|_1}{\||f|^{s/2}\|_2}\right)\right]_+^{1/r} \leqslant \|f\|_r \leqslant \|f\|_s\left[1-\frac{2(s-r)}{s}\left(1-\frac{\||f|^{s/2}\|_1}{\||f|^{s/2}\|_2}\right)\right]^{1/r}, \tag{3.1.1}$$

while if $s \geqslant 2r$, the inequalities hold with $2r/s$ and $2(s-r)/s$ interchanged.

Proof. We use Theorem 2.2 with $p = s/r > 1$, $|f|^r \in L^p$, $q = s/(s-r) > 1$ and $g \equiv 1$. Suppose first that $s \leqslant 2r$, i.e., that $1 < p \leqslant 2$. Substituting in (2.2.1) and simplifying we get (3.1.1). If $2 \leqslant p < \infty$ argue in the same way and use the last part of Theorem 2.2. □

More common measures of the dispersion of $|f|^{s/2}/\||f|^{s/2}\|_2$ around its mean are the variance Var and the standard deviation σ. From the previous result it is possible to derive bounds for $\|f\|_r$ in terms of $\operatorname{Var}(|f|^{s/2}/\||f|^{s/2}\|_2)$ and $\sigma(|f|^{s/2}/\||f|^{s/2}\|_2) = \sqrt{\operatorname{Var}(|f|^{s/2}/\||f|^{s/2}\|_2)}$.

Corollary 3.2. *Let $0 < r < s < \infty$, and suppose $0 < \|f\|_s < \infty$. If $s \leqslant 2r$, then*

$$\|f\|_s\left[1-\frac{2r}{s}\sigma\left(\frac{|f|^{s/2}}{\||f|^{s/2}\|_2}\right)\right]_+^{1/r} \leqslant \|f\|_r \leqslant \|f\|_s\left[1-\frac{s-r}{s}\operatorname{Var}\left(\frac{|f|^{s/2}}{\||f|^{s/2}\|_2}\right)\right]^{1/r}, \tag{3.2.1}$$

while if $s \geqslant 2r$, the same inequalities hold, but with the terms $2r/s$ and $(s-r)/s$ interchanged.

Proof. Note that for all $x \in [0, 1]$

$$2^{-1}\left(1-x^2\right) = 2^{-1}(1+x)(1-x) \leqslant (1-x) \leqslant \sqrt{1-x^2}, \tag{3.2.2}$$

where the last inequality can be checked simply by squaring both sides. Next we set $x = \||f|^{s/2}\|_1/\||f|^{s/2}\|_2$. Then $x \leqslant 1$ by either Jensen's inequality or more simply, the nonnegativity of the variance. Substituting in (3.2.2) we obtain

$$\frac{1}{2}\operatorname{Var}\left(\frac{|f|^{s/2}}{\||f|^{s/2}\|_2}\right) \leqslant \frac{\||f|^{s/2}\|_2 - \||f|^{s/2}\|_1}{\||f|^{s/2}\|_2} \leqslant \sigma\left(\frac{|f|^{s/2}}{\||f|^{s/2}\|_2}\right), \tag{3.2.3}$$

Now (3.2.1) follows from (3.1.1) when $s/r \leqslant 2$, while if $2 \leqslant s/r$, we use the last part of Theorem 3.1 to obtain the corresponding inequalities. □

Theorem 3.1 and Corollary 3.2 are stability results, in the sense that $\|f\|_s$ and $\|f\|_r$ are "close" if and only if $|f|$ is "nearly" constant; when $\angle(|f|^{s/2}, 1)$ (or $\operatorname{Var}(|f|^{s/2}/\||f|^{s/2}\|_2)$) is sufficiently small, these norms are comparable. We believe these results will be useful in contexts where information is available about the first and second moments of a function, as is often the case in probability theory.

Remark 3.3. It is easy to check that the factors between square brackets in the left-hand sides of (3.1.1) and (3.2.1) can actually be negative, so the positive part must be taken before raising them to the $1/r$ power. Take, for instance, $s = 2$, any fixed $r \in (1, 2)$, and $f = \sqrt{n}\chi_{[0,1/n]}$ on $[0, 1]$, with $n = n(r)$ "large enough."

A variant of the result on containment of L^p spaces exchanges the probability measure (or more generally, finite measure) hypothesis by the condition that f belongs to L^{p_0}, for some $p_0 < p$. We consider this next.

Theorem 3.4. *Let $0 < p_0 < p < p_1 < \infty$, and let $t = t(p)$ be given by the equation $p^{-1} = (1-t)p_0^{-1} + tp_1^{-1}$. Suppose $f \in L^{p_0} \cap L^{p_1}$ and $f \not\equiv 0$. If $p_0/p_1 \leqslant t^{-1} - 1$, then*

$$\|f\|_{p_0}^{1-t}\|f\|_{p_1}^{t}\left[1-\frac{2(1-t)p_1}{(1-t)p_1+tp_0}\left(1-\frac{\int|f|^{\frac{p_0+p_1}{2}}}{(\int|f|^{p_0})^{1/2}(\int|f|^{p_1})^{1/2}}\right)\right]_+^{1/p} \tag{3.4.1}$$

$$\leqslant \|f\|_p \leqslant \|f\|_{p_0}^{1-t}\|f\|_{p_1}^{t}\left[1-\frac{2tp_0}{(1-t)p_1+tp_0}\left(1-\frac{\int|f|^{\frac{p_0+p_1}{2}}}{(\int|f|^{p_0})^{1/2}(\int|f|^{p_1})^{1/2}}\right)\right]^{1/p}, \tag{3.4.2}$$

while if $p_0/p_1 \geqslant t^{-1}-1$, the inequalities are reversed, and the positive part of the term between square brackets is taken in the right-hand side of (3.4.2).

Proof. Again we use Theorem 2.2, with the functions $f^{(1-t)p}f^{tp} = f^p$, and the conjugate exponents $p_0/[(1-t)p]$ and p_1/tp. Note that $p_0/[(1-t)p] > 1$ and $p_1/tp > 1$, while $p_0/p_1 \leqslant t^{-1}-1$ if and only if $p_0/((1-t)p) \leqslant 2$. □

Remark 3.5. The preceding theorem leads to a midpoint interpolation result for arbitrary pairs of functions. Suppose, for instance, that $f, h \in L^{p_0} \cap L^{p_1}$, $f, h \not\equiv 0$, $\|f\|_{p_0} \leqslant \|h\|_{p_0}$, and $\|f\|_{p_1} \leqslant \|h\|_{p_1}$. It is easy to see that $\|f\|_p > \|h\|_p$ may happen for some intermediate $p \in (p_0, p_1)$. Consider the following example: Set $f(x) = (1-1/n)\chi_{[0,1/2]}$ on $[0,1]$, where $n \geqslant 6$ is fixed, and let $h(x) = x$. Then $\|f\|_1 < \|h\|_1$ and $\|f\|_\infty < \|h\|_\infty$, but $\|f\|_n > \|h\|_n$. Note that $\|f\|_p < \|h\|_p$ for every large enough $p < \infty$; in particular, if $n = 6$ we can take $p_1 = 11$, so there is a reversal of the inequality at $p = (p_0+p_1)/2$. However, under the additional condition on the angles $\angle(|h|^{p_0/2}, |h|^{p_1/2}) \leqslant \angle(|f|^{p_0/2}, |f|^{p_1/2})$, or equivalently, $\theta(|h|^{p_0/2}, |h|^{p_1/2}) \leqslant \theta(|f|^{p_0/2}, |f|^{p_1/2})$, at the midpoint $p = (p_0+p_1)/2$ we have $\|f\|_p \leqslant \|h\|_p$ whenever $\|f\|_{p_0} \leqslant \|h\|_{p_0}$ and $\|f\|_{p_1} \leqslant \|h\|_{p_1}$. To see this, note that if $p = (p_0+p_1)/2$, then $t = p_1/(p_0+p_1)$, so from (3.4.2) and (3.4.1) we get

$$\|f\|_p \leqslant \|f\|_{p_0}^{1-t}\|f\|_{p_1}^{t}\left[1-\frac{1}{2}\left\|\frac{|f|^{p_0/2}}{\||f|^{p_0/2}\|_2}-\frac{|f|^{p_1/2}}{\||f|^{p_1/2}\|_2}\right\|_2^2\right]^{1/p}$$

$$\leqslant \|h\|_{p_0}^{1-t}\|h\|_{p_1}^{t}\left[1-\frac{1}{2}\left\|\frac{|h|^{p_0/2}}{\||h|^{p_0/2}\|_2}-\frac{|h|^{p_1/2}}{\||h|^{p_1/2}\|_2}\right\|_2^2\right]^{1/p} \leqslant \|h\|_p.$$

Needless to say, stronger assumptions on the angles lead to stronger interpolation results. For instance, if $\theta(|h|^{p_0/2}, |h|^{p_1/2}) < \theta(|f|^{p_0/2}, |f|^{p_1/2})$, then $\|f\|_p < \|h\|_p$ for every p in some neighborhood of $(p_0+p_1)/2$, since the quantities involved in (3.4.2) and (3.4.1) change continuously. It is also possible to consider conditions of the type $\|f\|_{p_i} \leqslant c_i\|h\|_{p_i}$, with $c_i > 0$ not necessarily equal to 1, or even to have $h \in L^{r_0} \cap L^{r_1}$ with $r_i \neq p_i$, as is often done in interpolation theorems. But we will not pursue these elaborations here.

Remark 3.6. In standard interpolation results, such as the Riesz–Thorin and the Marcinkiewicz interpolation theorems, the pairing between the functions f and $h = T(f)$ is not arbitrary but given respectively by a linear or sublinear operator T, and the conclusion, of course, is much stronger than anything contained in the previous remark. The attentive reader may wonder why more general pairings are interesting, or in other words, whether there is any need to go beyond sublinearity. Next we give an example where such a result might be useful. It involves the derivative DMf of the one dimensional, uncentered Hardy–Littlewood maximal function Mf, defined as follows: Given a locally integrable function $f : \mathbb{R} \to \mathbb{R}$,

$$Mf(x) := \sup_{x \in I} \frac{1}{|I|}\int_I |f(y)|\,dy,$$

where I is any interval containing x and $|I|$ stands for its length. Starting with the paper [13], there has been in recent years a growing interest regarding the regularity of the maximal function (cf., for instance, [2] and the references contained therein). Suppose for simplicity that $f : \mathbb{R} \to \mathbb{R}$ is a compactly supported Lipschitz function. It is shown in [13] (cf. also [12]) that for every $1 < p \leqslant \infty$ there is a constant c_p (independent of f) such that $\|DMf\|_p \leqslant c_p\|Df\|_p$. However, the methods used in [13] and [12] cannot tell us whether we actually have $c_p < 1$, that is, whether the maximal operator M has a smoothing effect on f. For $p = 1$, Theorem 2.5 of [2] states that $\|DMf\|_1 \leqslant \|Df\|_1$, and $c_1 = 1$ is sharp, while for $p = \infty$, we have $\|DMf\|_\infty \leqslant (\sqrt{2}-1)\|Df\|_\infty$ and $c_\infty = (\sqrt{2}-1)$ is best possible, by [1]. Thus, it is natural to conjecture "by interpolation" that whenever $1 < p < \infty$, the optimal constant c_p satisfies $c_p < 1$, and furthermore, $\lim_{p\to\infty} c_p = \sqrt{2}-1$. Nevertheless, since the operator $Df \mapsto DMf$ is neither linear nor sublinear, it falls outside the realm of currently available interpolation theorems. Unfortunately, the second endpoint for which information is available happens to be $p = \infty$, so our stability version of Hölder's inequality also fails to yield anything new on this question.

4. The triangle inequality and uniform convexity

Like Clarkson's inequalities and Hanner's inequalities, formula (2.2.1) can lay claim to being an L^p generalization of the parallelogram identity. Furthermore, despite its easy proof, the refinement of Hölder's inequality presented above does have strength: It gives, by sharpening Minkowski's inequality, the uniform convexity of L^p spaces (at least in the real valued case), with the right asymptotic behavior of the modulus of convexity for all $p \in (1, \infty)$. The exact asymptotic behavior was found by O. Hanner (cf. [10], or [14, p. 63]); Clarkson's original inequalities (see the corollary in [5, p. 403]) yield it over the range $2 \leqslant p < \infty$, but not for $1 < p < 2$.

The arguments presented here only cover the real valued case, and the complex valued case if $p \geqslant 2$. Since only the moduli of functions (and not their signs) play any role in the sizes of $\|fg\|_1$ and $\|f\|_p\|g\|_q$, the same must necessarily happen with the error terms in any refinement of Hölder's inequality. In particular, this is the case with (2.2.1). But for some applications, such as a refined triangle inequality, it would be preferable to control the departure from maximal size in terms of $|f - g|$ rather than $||f| - |g||$. We shall show that for real valued functions, and for complex valued functions when $p \geqslant 2$, one can assume the comparability of $\|f - g\|_p$ and $\||f| - |g|\|_p$. But the proof in the complex case when $1 < p < 2$ has eluded us. A recent, new proof of uniform convexity, relying on the notion of thin slices and which does apply to the complex case, can be found in [11] (however, there the author is unconcerned about the precise behavior of the modulus of convexity).

The improved Minkowski's inequality given next is obtained from our refinement of Hölder's inequality by the usual "duality" argument. By the "duality" argument we do not mean knowing that the dual of L^p is L^q, but simply that

$$\|f\|_p = \sup_{\{g \in L^q:\ \|g\|_q = 1\}} \int fg, \tag{4.0.1}$$

which follows from Hölder's inequality together with the trivial observation that equality is achieved when $g = \frac{|f|^{p-1}\overline{\operatorname{sign} f}}{\|f\|_p^{p-1}}$. Here $\operatorname{sign}(z) := e^{i\theta}$ for every complex nonzero $z = re^{i\theta}$, and $\operatorname{sign}(0) := 1$ (we adopt this convention, rather than the usual $\operatorname{sign}(0) := 0$, since in order to multiply quantities without changing sizes it is useful to always have $|\operatorname{sign}(z)| = 1$). As is well known, (4.0.1) immediately entails the triangle inequality:

$$\|f + h\|_p = \sup_{\{g \in L^q:\ \|g\|_q = 1\}} \int (f + h)g \leqslant \sup_{\{g_1 \in L^q:\ \|g_1\|_q = 1\}} \int fg_1 + \sup_{\{g_2 \in L^q:\ \|g_2\|_q = 1\}} \int hg_2 = \|f\|_p + \|h\|_p. \tag{4.0.2}$$

However, usually this proof appears with the explicit maximizing g written in place of the first supremum, and then it proceeds from there. As it turns out, it will be more convenient for us to do likewise below.

Theorem 4.1. *Let $1 < p < \infty$. If $f, h \in L^p$, $\|f\|_p, \|h\|_p > 0$, and $1 < p \leqslant 2$, then*

$$\|f + h\|_p \leqslant \|f\|_p \left(1 - \frac{1}{q}\left\|\frac{|f + h|^{p/2}}{\|f + h\|_p^{p/2}} - \frac{|f|^{p/2}}{\|f\|_p^{p/2}}\right\|_2^2\right) + \|h\|_p \left(1 - \frac{1}{q}\left\|\frac{|f + h|^{p/2}}{\|f + h\|_p^{p/2}} - \frac{|h|^{p/2}}{\|h\|_p^{p/2}}\right\|_2^2\right), \tag{4.1.1}$$

while if $2 \leqslant p < \infty$ the same inequality holds, but with $1/p$ replacing $1/q$ throughout.

Proof. Suppose $1 < p \leqslant 2$. Then

$$\|f + h\|_p = \int \frac{|f + h|^{p-1}}{\||f + h|^{p-1}\|_q}|f + h| \leqslant \int \frac{|f + h|^{p-1}}{\||f + h|^{p-1}\|_q}|f| + \int \frac{|f + h|^{p-1}}{\||f + h|^{p-1}\|_q}|h| \tag{4.1.2}$$

and the result follows by applying (2.2.1). If $2 \leqslant p < \infty$ argue in the same way and use the last part of Theorem 2.2. □

Next, we recall some basic facts about the Mazur map $\psi_{r,s} : L^r \to L^s$. It is defined first on the unit sphere by $\psi_{r,s}(f) := |f|^{r/s} \operatorname{sign} f$, and then extended to the rest of the space by homogeneity (cf. [4, pp. 197–199] for additional information on $\psi_{r,s}$). The "angle" $\| \frac{|f|^{p/2}}{\|f\|_p^{p/2}} - \frac{|g|^{q/2}}{\|g\|_q^{q/2}} \|_2$ in (2.2.1) is obtained by applying the Mazur maps from the nonnegative functions in the unit spheres of L^p and L^q, into the unit sphere of L^2. Thus, we have control over the distortion, since when $r < s$, the map $\psi_{s,r}$ is Lipschitz on the unit sphere of L^s, with constant s/r, while its inverse $\psi_{r,s}$ is Hölder with exponent r/s. This is the content of the following well-known lemma, included here for the reader's convenience. It is a special case of Proposition 9.2 of [4, pp. 198–199], cf. also the proof of Theorem 9.1, p. 198, partially sketched below. Note however that in [4] the harder, complex valued case is handled, and the Hölder constant (as opposed to the Hölder exponent) is not specified. We will consider the Mazur map acting only on nonnegative functions, since that is all we shall use. In this easy case we show that the Hölder constant is 1.

Lemma 4.2. *Let $1 < r < s < \infty$, and let $f, h \geqslant 0$. If $f, h \in L^r$ satisfy $\|f\|_r = \|h\|_r = 1$, then $\|f^{r/s} - h^{r/s}\|_s \leqslant \|f - h\|_r^{r/s}$, while if $f, h \in L^s$ have norms $\|f\|_s = \|h\|_s = 1$, then $\|f^{s/r} - h^{s/r}\|_r \leqslant (s/r)\|f - h\|_s$.*

Proof. To prove the Hölder assertion, note that by concavity of t^α for $0 < \alpha < 1$, if $a > b$, then $a^\alpha - b^\alpha \leqslant (a - b)^\alpha$. Suppose f and h are nonnegative functions of norm 1 in L^r. Taking $\alpha = r/s$ and integrating the pointwise inequality $|f^{r/s}(x) - h^{r/s}(x)|^s \leqslant |f(x) - h(x)|^r$ we get $\|f^{r/s} - h^{r/s}\|_s \leqslant \|f - h\|_r^{r/s}$.

We sketch the proof the Lipschitz claim, directing the reader to [4] for additional details. Let us denote by $d\psi_{s,r}(f)(h)$ the Gateaux (i.e., the directional) derivative of the Mazur map based at the point f and in the direction of h, where the nonnegative functions f and h belong the unit sphere of L^s. It is enough to show that $\|d\psi_{s,r}(f)(h)\|_r^r \leqslant (s/r)^r$, which follows by explicit computation of the directional derivative, and an application of Hölder's inequality together with $\|f\|_s = \|h\|_s = 1$. □

After proving a simple lemma, we use the properties of the Mazur map to express the preceding triangle inequality in terms of the p norm.

Lemma 4.3. *Let x, y, z be vectors in a normed space, and let $p \in (1, \infty)$. Then $\|x - y\|^p \leqslant 2^{p-1}(\|x - z\|^p + \|y - z\|^p)$.*

Proof. We may assume that $x \neq y$. Since $\|x - y\| \leqslant \|x - z\| + \|y - z\|$, writing $a := \|x - z\|/\|x - y\|$ and $b := \|y - z\|/\|x - y\|$ we have that $a + b \geqslant 1$ and $(a^p + b^p)\|x - y\|^p = \|x - z\|^p + \|y - z\|^p$. Minimizing $a^p + b^p$ subject to $a + b \geqslant 1$ and $a, b \geqslant 0$, we obtain $a = b = 1/2$, from which the result follows. □

Clarkson gave asymptotic estimates for the modulus of convexity of L^p of order $O(\varepsilon^p)$ when $2 \leqslant p < \infty$ and $O(\varepsilon^q)$ when $1 < p \leqslant 2$, where $\varepsilon = \|f - h\|_p$. The optimal estimate $O(\varepsilon^2)$ when $1 < p \leqslant 2$ was found by Hanner. It is easy for us to explain this different behavior in terms of the Mazur map: When $p \leqslant 2$ the map $\psi_{2,p}$ is Lipschitz, and hence the exponent 2 in the error term from (2.0.2) or (2.2.1) is preserved, while if $p \geqslant 2$, then $\psi_{2,p}$ is $2/p$-Hölder, so the exponent 2 changes to p.

Corollary 4.4. *Let $1 < p < \infty$, and let $f, h \in L^p$. If $1 < p \leqslant 2$, then*

$$\|f + h\|_p \leqslant \|f\|_p + \|h\|_p - \min\{\|f\|_p, \|h\|_p\}\left(\frac{p(p-1)}{8}\left\|\frac{|f|}{\|f\|_p} - \frac{|h|}{\|h\|_p}\right\|_p^2\right), \tag{4.4.1}$$

while if $2 \leqslant p < \infty$,

$$\|f + h\|_p \leqslant \|f\|_p + \|h\|_p - \min\{\|f\|_p, \|h\|_p\}\left(\frac{1}{2p}\left\|\frac{|f|}{\|f\|_p} - \frac{|h|}{\|h\|_p}\right\|_p^p\right). \tag{4.4.2}$$

Proof. The result follows from Theorem 4.1, the previous lemma, and Lemma 4.2. □

Suppose, in order to simplify the corresponding expressions, that $\|f\|_p = \|h\|_p = 1$. A drawback of the preceding corollary is that in the right-hand side we have $\||f| - |h|\|_p$ rather than $\|f - h\|_p$, while the left-hand side depends on $f+h$, not on $|f|+|h|$. This is unavoidable since we are deriving the result from the stability version of Hölder's inequality (2.2.1). Thus, the case where $\||f|-|h|\|_p \ll \|f-h\|_p$ must be handled via a separate argument, which somehow we have failed to find when f and h are complex valued and $p<2$. The real valued case is easy since the only possibility for cancellation is to have opposite signs, and for $p \geqslant 2$ the complex valued case immediately follows from the convexity of $p/2$.

Note that the bound in the next proposition has nothing to do with uniform convexity: It holds even when $p=1$. In fact, all we are doing is checking the intuitively obvious fact that if we want $\|f+h\|_p$ to be large, the signs of f and h must be very similar, specially if p is small. While this ought to be also true in the complex valued case, as I said I have not been able to prove it.

Proposition 4.5. *Let $1 \leqslant p < \infty$, let $0<t<1$, and let $f,h \in L^p$ be real valued functions. If $\||f|-|h|\|_p^p < t\|f-h\|_p^p$, then $\|f+h\|_p < ((\|f\|_p + \|h\|_p)^p - (1-t)\|f-h\|_p^p)^{1/p}$.*

Proof. First, we may assume that $f \geqslant 0$, since by the convention sign(0) := 1 (adopted just after (4.0.1)) given any x we have $|f(x)-h(x)| = |f(x)\operatorname{sign} f(x) - h(x)\operatorname{sign} f(x)|$, and likewise for $|f(x)+h(x)|$. Next, note that if $a \geqslant 0$ and $b \in \mathbb{R}$, then $|a+b|^p + |a-b|^p = |a+|b||^p + |a-|b||^p$, so writing $f(x)=a$, $h(x)=b$, and integrating, we get

$$\|f+h\|_p^p = \bigl\|f+|h|\bigr\|_p^p + \bigl\|f-|h|\bigr\|_p^p - \|f-h\|_p^p \leqslant \bigl(\|f\|_p + \|h\|_p\bigr)^p - (1-t)\|f-h\|_p^p. \qquad \square \tag{4.5.1}$$

Remark 4.6. Note that by Taylor's formula (or by linear approximation at 0 and concavity), we have $(1-x)^{1/p} \leqslant 1-p^{-1}x$. Applying this inequality to the conclusion of the previous proposition when $\|f\|_p = \|h\|_p = 1$, we get

$$\left\|\frac{f+h}{2}\right\|_p \leqslant 1 - \frac{1-t}{p2^p}\|f-h\|_p^p. \tag{4.6.1}$$

Let B be a Banach space. Clarkson's original definition of uniform convexity requires that for every $0<\varepsilon\leqslant 2$ there exists $\delta(\varepsilon)>0$ such that if $\|f\| = \|h\| = 1$ and $\|f-h\| \geqslant \varepsilon$, then $\|\frac{f+h}{2}\| \leqslant 1-\delta(\varepsilon)$ (cf. [5, Definition 1, pp. 396–397]). The often used and seemingly weaker assumption $\|f\|, \|h\| \leqslant 1$ is of course equivalent to $\|f\| = \|h\| = 1$, since f and h must have norm one in order to maximize $\|f+h\|_p$ subject to $\|f-h\| \geqslant \varepsilon$ (see [6, Lemma 5.1, p. 381] for a full proof). In the words of [3], B is uniformly convex if its unit ball is "uniformly free of flat spots." From the viewpoint of the geometry of B is often interesting to have a good estimate of how δ depends on $\varepsilon \in (0,2]$. The following definitions and results are taken from [14, specially p. 63]. The modulus of convexity δ_B of B is given by

$$\delta_B(\varepsilon) := \inf\left\{1 - \left\|\frac{f+h}{2}\right\|:\ \|f\| = \|h\| = 1,\ \|f-h\| = \varepsilon\right\}. \tag{4.6.2}$$

We say that δ_B is of power type r if there exists a constant $c>0$ such that $\delta_B(\varepsilon) \geqslant c\varepsilon^r$. For $B=L^p$ and $1<p\leqslant 2$, $\delta_B(\varepsilon) = (p-1)\varepsilon^2/8 + o(\varepsilon^2)$, while for $2 \leqslant p < \infty$, $\delta_B(\varepsilon) = \varepsilon^p/(p2^p) + o(\varepsilon^p)$.

The next result shows that in the real valued case, the preceding variants of the triangle inequality yield the optimal value of r in the power type estimates. The constants, however, are not optimal. But they are not too far away from optimality either. We make an effort to obtain "fairly good" constants for the modulus of convexity (and not just good power type estimates, which is all one usually needs for applications) since this entails that the constants in the original inequality (2.2.1) must also be "fairly good."

Theorem 4.7. *Let $1<p<\infty$. Then $B = L^p(X,\mathbb{R})$ is uniformly convex. Furthermore, its modulus of convexity satisfies the following inequalities. If $p \in (1,2]$, then for every $c>1$ there exists $\varepsilon = \varepsilon(c)$ such that for all $f,h \in L^p$ with $\|f\|_p = \|h\|_p = 1$ and $\|f-h\|_p \leqslant \varepsilon$,*

$$\delta_B\bigl(\|f-h\|_p\bigr) \geqslant \frac{p(p-1)}{16c}\|f-h\|_p^2. \tag{4.7.1}$$

On the other hand, if $2 \leqslant p < \infty$, then for all $f,h \in L^p$ with $\|f\|_p = \|h\|_p = 1$,

$$\delta_B\bigl(\|f-h\|_p\bigr) \geqslant \frac{\|f-h\|_p^p}{p2^p + 4p}. \tag{4.7.2}$$

Proof. Note that for every $p \in (1, \infty)$ and every $t \in (0, 1)$, if $\||f| - |h|\|_p^p < t\|f - h\|_p^p$, then

$$\delta_B(\|f - h\|_p) \geqslant \frac{(1-t)\|f - h\|_p^p}{p2^p} \tag{4.7.3}$$

by Proposition 4.5, or more precisely, by (4.6.1).

We prove (4.7.2) first. Given $t \in (0, 1)$, if $\||f| - |h|\|_p^p \geqslant t\|f - h\|_p^p$, by (4.4.2) we have the bound

$$\delta_B(\|f - h\|_p) \geqslant \frac{t\|f - h\|_p^p}{4p}. \tag{4.7.4}$$

Choosing t, so that the lower bounds given by (4.7.3) and (4.7.4) are equal, (4.7.2) follows.

With respect to (4.7.1), observe that for every $t \in (0, 1)$ and $\|f - h\|_p$ sufficiently small (depending on t), the bound

$$\delta_B(\|f - h\|_p) \geqslant \frac{t^{2/p} p(p-1)\|f - h\|_p^2}{16}, \tag{4.7.5}$$

which follows from (4.4.1) when $\||f| - |h|\|_p^p \geqslant t\|f - h\|_p^p$, is always smaller than the bound given by (4.7.3) when $\||f| - |h|\|_p^p < t\|f - h\|_p^p$. Writing $c = t^{-2/p}$, (4.7.1) follows by fixing $\varepsilon > 0$ small enough and taking $\|f - h\|_p \leqslant \varepsilon$. □

We have given an asymptotic estimate when $1 < p \leqslant 2$ in order to be as precise as we can. If we are not concerned with good constants, to obtain a statement which does not require ε to be small we can just fix any t (say $t = 2^{-1}$ for definiteness) and take the minimum of the quantities given by (4.7.3) and (4.7.5).

Next we consider the case of $L^p(X, \mathbb{C})$ spaces, when $p \geqslant 2$. The argument is essentially the same as in Proposition 4.5.

Proposition 4.8. *Let $2 \leqslant p < \infty$, let $0 < t < 1$, and let $f, h \in L^p$ be complex valued functions. If $\||f| - |h|\|_p^p < t\|f - h\|_p^p$, then $\|f + h\|_p < ((\|f\|_p + \|h\|_p)^p - (1-t)\|f - h\|_p^p)^{1/p}$.*

Proof. As before, we may assume that $f \geqslant 0$. Writing $h = |h|e^{i\alpha}$, where $\alpha = \alpha(h(x))$, we have that for every x,

$$\begin{aligned} |f(x) + h(x)|^p + |f(x) - h(x)|^p &= |f^2(x) + |h(x)|^2 + 2f(x)|h(x)|\cos\alpha(h(x))|^{p/2} \\ &\quad + |f^2(x) + |h(x)|^2 - 2f(x)|h(x)|\cos\alpha(h(x))|^{p/2}. \end{aligned} \tag{4.8.1}$$

By the convexity of $t^{p/2}$,

$$\begin{aligned} &|f(x) + h(x)|^p + |f(x) - h(x)|^p \\ &\quad \leqslant |f^2(x) + |h(x)|^2 + 2|h(x)|f(x)|^{p/2} + |f^2(x) + |h(x)|^2 - 2|h(x)|f(x)|^{p/2} \\ &\quad = |f(x) + |h|(x)|^p + |f(x) - |h|(x)|^p. \end{aligned} \tag{4.8.2}$$

The rest of the proof is as in Proposition 4.5. □

Remark 4.9. From the preceding proposition and the second part of Corollary 4.4, the uniform convexity of the $L^p(X, \mathbb{C})$ spaces when $p \geqslant 2$ follows in exactly the same way and with the same constants as in Theorem 4.7, so we avoid the repetition.

Remark 4.10. As we have noted, a disadvantage of the refined triangle inequality given in Corollary 4.4, is that the error or stability term depends only on the moduli of the functions involved, and not their signs. But this inequality has its advantages also. One of them is that it interacts well with other inequalities given here, in the sense that it is easy to obtain nontrivial information by combining them. For instance, suppose $\mu(X) = 1$ and $0 < r < s$, with $f, h \in L^s$. Under suitable hypotheses on the variance of $|f + h|^{s/2}$, we can easily find bounds for $\|f + h\|_s$ in terms of $\|f\|_r$ and $\|h\|_r$, by using Theorem 3.1 or Corollary 3.2, together with Corollary 4.4. Alternatively, we might be interested, say, in bounding $\|f + h\|_r$ in terms of $\|f\|_s$ and $\|h\|_s$. Thus, there are several possibilities to study the behavior of $\|f + h\|_p$ as p changes.

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