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J. Math. Anal. Appl. 329 (2007) 1170–1180

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Weighted inequalities for the Bochner–Riesz means related to the Fourier–Bessel expansions

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Received 14 February 2006

Available online 10 August 2006

Submitted by D. Waterman

Abstract

We prove weighted inequalities for the Bochner–Riesz means for Fourier–Bessel series with more general weights $w(x)$ than previously considered power weights. These estimates are given by using the local A_p theory and Hardy's inequalities with weights. Moreover, we also obtain weighted weak type $(1,1)$ inequalities. The case when $w(x)=x^a$ is sketched and follows as a corollary of the main result.

Keywords: Fourier–Bessel series; Bochner–Riesz means; Two-weight inequalities; Weak type inequalities

1. Introduction

Given $\nu>-1$, let $\{s_j\}_{j\geqslant 1}$ denote the sequence of successive positive zeros of the Bessel function J_ν. Then the functions

$$\phi_j(x)=\frac{\sqrt{2x}\,J_\nu(s_jx)}{|J_{\nu+1}(s_j)|},\quad j=1,2,\dots,$$

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[1] Research of the author supported by grant MTM2006-13000-C03-03 of the D.G.I.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.07.014

form a complete orthonormal system in $L^2((0,1),dx)$ (for the completeness, see [5]). The Fourier–Bessel expansion of a function f is

$$\sum_{j=1}^{\infty}\phi_j(x)\left(\int_0^1 f(y)\phi_j(y)\,dy\right),$$

provided the integrals exist. For an extensive study about Fourier–Bessel series, see [9, Chapter XVII]. For $\delta>0$ the Bochner–Riesz means of this expansion are

$$B_R^\delta(f,x)=\sum_{j\geqslant 1}\left(1-\frac{s_j^2}{R^2}\right)_+^\delta\phi_j(x)\left(\int_0^1 f(y)\phi_j(y)\,dy\right),$$

where $R>0$ and $(1-s^2)_+=\max\{1-s^2,0\}$. It follows that

$$B_R^\delta(f,x)=\int_0^1 f(y)K_R^\delta(x,y)\,dy,$$

where

$$K_R^\delta(x,y)=\sum_{j\geqslant 1}\left(1-\frac{s_j^2}{R^2}\right)_+^\delta\phi_j(x)\phi_j(y).$$

In [2] we proved the mean convergence of this summation method, and in [3] we showed results about almost everywhere convergence. In both cases we analyzed the results in Lebesgue spaces with power weights. Specifically, the mean convergence for the operator B_R^δ follows from the weighted estimate, for $1\leqslant p<\infty$,

$$\left\|x^a B_R^\delta(f,x)\right\|_{L^p}\leqslant C\left\|x^A f(x)\right\|_{L^p}, \tag{1}$$

where $\|g\|_{L^p}$ expresses unweighted L^p norm of g on $(0,1)$ with C independent of R and g, under certain conditions for a, A, ν and δ. The conditions given in [2] for (1) are necessary and sufficient. Then, the convergence of the Bochner–Riesz means is deduced by the density of the orthonormal system considered.

In the present paper, we use several techniques that are different from those introduced in [2], in order to include more general weights. To be precise, we deal with the local A_p theory and Hardy's inequalities with weights. In addition to that, our new approach also gives weighted weak type $(1,1)$ inequalities.

Hardy's inequalities with weights and weighted weak type Hardy's inequalities were discussed in [6] and [1], respectively, and we use those results here. Moreover, we need the theory about local A_p developed in [8].

Other works study weighted estimates with general weights using this kind of ideas; the most relevant are those concerning transplantation operators. In [8] Nowak and Stempak proved weighted estimates results for the Hankel transform transplantation operator. Since the Fourier–Bessel expansions can be seen as the discrete analogue of the Hankel transform, it is also interesting to note that the approach developed in [8] leads to a natural approach for the discrete case. Then, this case is studied in [4]. In both of them, the transplantation operator is investigated by means of a suitable local version of the Calderón–Zygmund operator theory. In our paper, we consider the Bochner–Riesz means for the Fourier–Bessel expansions; we prove weighted inequalities with weights more general than power weights by using the tools described before. Besides, we also obtain weighted weak type (1, 1) inequalities. In this setting, we consider the local version of the Hardy–Littlewood maximal function instead of the local version of the Calderón–Zygmund theory.

For the kernel associated with the operator B_R^δ we will consider the decomposition

$$\left|K_R^\delta(x,y)\right| = \sum_{k=1}^{5}\left|K_R^\delta(x,y)\right|\chi_{A_k}(x,y),$$

where

$$A_1 = \left\{(x,y)\colon\ 0 < x, y \leqslant \frac{4}{R}\right\},$$
$$A_2 = \left\{(x,y)\colon\ \frac{4}{R} < \max\{x,y\} < 1,\ |x-y| \leqslant \frac{4}{3R}\right\},$$
$$A_3 = \left\{(x,y)\colon\ \frac{4}{R} \leqslant x < 1,\ 0 < y \leqslant \frac{x}{2}\right\},$$
$$A_4 = \left\{(x,y)\colon\ 0 < x \leqslant \frac{2y}{3},\ \frac{4}{R} \leqslant y < 1\right\},$$
$$A_5 = \left\{(x,y)\colon\ \frac{4}{R} < x < 1,\ \frac{x}{2} < y < x - \frac{4}{3R}\right\} \cup \left\{(x,y)\colon\ \frac{y}{2} < x \leqslant y - \frac{4}{3R},\ \frac{4}{R} \leqslant y < 1\right\}.$$

Then, $|B_R^\delta f| \leqslant \sum_{k=1}^5 T_k f$, where we define

$$T_k(f,x) = \int_0^1 f(y)\left|K_R^\delta(x,y)\right|\chi_{A_k}(x,y)\,dy.$$

The operators T_3 and T_4 will be analyzed by means of weighted Hardy's inequalities. To treat T_2 and T_5 we apply the local A_p theory. At last, T_1 is readily handled by standard facts. To analyze these operators, we must take into account the estimates obtained for the kernel K_R^δ in [2]. The regions considered in [2] are not exactly the previously given ones but they are similar and the estimates in that paper imply, without additional effort, that

$$\left|K_R^\delta(x,y)\right| \leqslant C \begin{cases} (xy)^{\nu+1/2}R^{2(\nu+1)}, & (x,y) \in A_1,\\ R, & (x,y) \in A_2,\\ \frac{\Phi_\nu(Rx)\Phi_\nu(Ry)}{R^\delta|x-y|^{\delta+1}}, & (x,y) \in A_3 \cup A_4 \cup A_5, \end{cases} \tag{2}$$

with

$$\Phi_\nu(t) = \begin{cases} t^{\nu+1/2}, & \text{if } 0 < t < 4,\\ 1, & \text{if } t \geqslant 4. \end{cases}$$

Throughout the paper we use the following standard notation. Thus, for a weight w on $(0,1)$ (a nonnegative measurable function such that $w(x) < \infty$ x-a.e.) we use $L^p(w)$ and $L^{1,\infty}(w)$ to denote the weighted L^p and the weighted weak L^1 spaces (with respect to the Lebesgue measure dx) that consist of all functions f on $(0,1)$ for which

$$\|f\|_{L^p(w)} = \left(\int_0^1 \left|f(x)w(x)\right|^p dx\right)^{1/p} < \infty$$

or

$$\|f\|_{L^{1,\infty}(w)} = \sup_{\lambda>0}\left(\lambda \int_{\{0<x<1:\ |f(x)|>\lambda\}} w(x)\,dx\right) < \infty,$$

respectively. If $w \equiv 1$, we simplify the notation by writing L^p or $L^{1,\infty}$. Given $1 \leqslant p \leqslant \infty$, p' denotes its conjugate, $1/p + 1/p' = 1$.

The structure of the paper is as follows. In Section 2 we state the main results of the paper; they are contained in Theorems 1 and 2. Section 3 is focused on the proofs of these results.

2. Statement of results

Given a weight function $u(x)$ on $(0,1)$, consider the following set of conditions:

$$\sup_{0<R}\sup_{0<r<1}\left(\int_r^1 \left(\frac{u(x)\chi_{[4/R,1)}(x)}{R^\delta x^{\delta+1}}\right)^p dx\right)^{1/p}\left(\int_0^r \left(\frac{\Phi_\nu(Rx)}{u(x)}\right)^{p'} dx\right)^{1/p'} < \infty, \tag{3}$$

$$\sup_{0<R}\sup_{0<r<1}\left(\int_0^r \big(u(x)\Phi_\nu(Rx)\big)^p\,dx\right)^{1/p}\left(\int_r^1 \left(\frac{\chi_{[4/R,1)}(x)}{u(x)R^\delta x^{(\delta+1)}}\right)^{p'} dx\right)^{1/p'} < \infty, \tag{4}$$

$$\sup_{0<w<v<\min\{1,2w\}} \frac{1}{v-w}\left(\int_w^v u(x)^p\,dx\right)^{1/p}\left(\int_w^v u(x)^{-p'}\,dx\right)^{1/p'} < \infty, \tag{5}$$

$$\sup_{0<R} R^{2(\nu+1)}\left(\int_0^{4/R} \big(u(x)x^{\nu+1/2}\big)^p\,dx\right)^{1/p}\left(\int_0^{4/R}\left(\frac{x^{\nu+1/2}}{u(x)}\right)^{p'} dx\right)^{1/p'} < \infty. \tag{6}$$

For a weight u satisfying (5) we write $u^p \in A_{p,\mathrm{loc}}(0,1)$ and say that u^p is a local A_p weight. The left side of (5) is then called the $A_{p,\mathrm{loc}}$ norm of u^p. We admit $1 \leqslant p < \infty$ when considering conditions (3)–(6). From now on, for $p' = \infty$ integrals of the form appearing in (3)–(6) have the usual interpretation. For example, the second factor in (3) is taken as $\operatorname{ess\,sup}_{x\in(0,r)}[\Phi_\nu(Rx)u(x)^{-1}]$.

The condition (3) is sufficient for the weighted Hardy's inequality

$$\left(\int_0^1 \left|\frac{u(x)\chi_{[4/R,1)}(x)}{R^\delta x^{\delta+1}}\int_0^x f(t)\,dt\right|^p dx\right)^{1/p} \leqslant C\left(\int_0^1 \left|\frac{u(x)}{\Phi_\nu(Rx)}f(x)\right|^p dx\right)^{1/p} \tag{7}$$

to hold with a constant C independent of R, while the condition (4) is sufficient for its dual version

$$\left(\int_0^1 \left|u(x)\Phi_\nu(Rx)\int_x^1 f(t)\,dt\right|^p dx\right)^{1/p}$$
$$\leqslant C\left(\int_0^1 \left|u(x)R^\delta x^{(\delta+1)}\chi_{[4/R,1)}(x)f(x)\right|^p dx\right)^{1/p} \tag{8}$$

to be satisfied, also with the constant C not depending on R. These two facts follow from a modification of Theorems 1 and 2 in [6]. The conditions on the weights to obtain the inequalities for the Hardy's operators in [6] do not involve the supremum on R, with this inclusion in (3) and (4) we reach uniform weighted inequalities in (7) and (8). The local A_p condition (5) for u^p is, for $1 < p < \infty$, necessary and sufficient for the estimate

$$\int_0^1 \left|M(f,x)u(x)\right|^p dx \leqslant \int_0^1 \left|f(x)u(x)\right|^p dx \tag{9}$$

to hold, where M denotes the local version of the one-dimensional Hardy–Littlewood maximal operator

$$M(f,x) = \sup_{|x-y|\leqslant x/2} \frac{1}{y-x} \int_x^y \left|f(t)\right| dt, \quad x \in (0,1).$$

Sufficiency part above is just a version of [7, Lemma 9.6, p. 30]. Necessity of (5) is provided in [8, Section 6]. In the case $p = 1$ condition (5) is necessary and sufficient for the weighted weak type $(1,1)$ inequality

$$\int_{\{0<x<1:\ |M(f,x)|>\lambda\}} u(x) \leqslant \frac{C}{\lambda} \int_0^1 \left|f(x)\right|u(x)\, dx, \quad \lambda > 0, \tag{10}$$

to hold, see [8, Section 6]. Finally, condition (6) is a technical requirement that we need to estimate our operator in the square A_1.

The main result of the paper is contained in the following

Theorem 1. *Let $\nu > -1$, $\delta > 0$, $1 < p < \infty$ and $R > 0$. Let $u(x)$ be a weight that satisfies the conditions* (3)–(6). *Then*

$$\left\|B_R^\delta f\right\|_{L^p(u)} \leqslant C\|f\|_{L^p(u)} \tag{11}$$

for all $f \in L^p(u)$, with a constant C independent of R and f.

Corollary 1. *Let $\nu > -1$, $\delta > 0$, $1 < p < \infty$ and $R > 0$. Then*

$$\left\|B_R^\delta f\right\|_{L^p(x^a)} \leqslant C\|f\|_{L^p(x^a)}$$

if and only if

$$-1/p - (\nu + 1/2) < a < 1 - 1/p + (\nu + 1/2), \tag{12}$$

$$-\delta - 1/p < a < 1 + \delta - 1/p. \tag{13}$$

In order to treat weighted weak type $(1,1)$ inequalities for Bochner–Riesz means, for a given weight function $u(x)$ on $(0,1)$, consider the following set of conditions:

$$\sup_{0<R} \sup_{0<r<1} \left(\int_r^1 \left(\frac{r}{x}\right)^\alpha \frac{u(x)\chi_{[4/R,1)}(x)}{R^\delta x^{\delta+1}}\, dx \right) \left(\operatorname*{ess\,sup}_{x\in(0,r)} \frac{\Phi_\nu(Rx)}{u(x)} \right) < \infty, \tag{14}$$

$$\sup_{0<R}\sup_{0<r<1} r^{\nu+1/2}\left(\int_0^r u(x)\chi_{(0,4/R)}(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in(r,1)}\frac{R^{\nu+1/2-\delta}\chi_{[4/R,1)}(x)}{x^{\delta+1}u(x)}\right)<\infty, \tag{15}$$

$$\sup_{0<R}\sup_{0<r<1}\left(\int_0^r\left(\frac{x}{r}\right)^{\alpha}u(x)x^{\nu+1/2}\chi_{(0,4/R)}(x)\,dx\right)$$
$$\times\left(\operatorname*{ess\,sup}_{x\in(r,1)}\frac{R^{\nu+1/2-\delta}\chi_{[4/R,1)}(x)}{x^{\delta+1}u(x)}\right)<\infty, \tag{16}$$

$$\sup_{0<R}\sup_{0<r<1}\left(\int_0^r u(x)\chi_{[4/R,1)}(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in(r,1)}\frac{\chi_{[4/R,1)}(x)}{R^{\delta}x^{\delta+1}u(x)}\right)<\infty, \tag{17}$$

$$\sup_{0<R} R^{2(\nu+1)}\left\|x^{\nu+1/2}\chi_{(0,4/R)}(x)\right\|_{L^{1,\infty}(u)}\sup_{0<x<4/R}\frac{x^{\nu+1/2}}{u(x)}<\infty. \tag{18}$$

In (14) and (16) we assume that there exists a positive α such that the quantity is finite. Let P_η, Q_η, η real, denote the Hardy operators acting on functions defined on $(0,1)$:

$$P_\eta(f,x)=x^{-\eta}\int_0^x f(t)\,dt,\qquad Q_\eta(f,x)=x^{-\eta}\int_x^1 f(t)\,dt,\qquad 0<x<1.$$

The condition (14) is sufficient for the inequality

$$\int_{\{0<x<1:\ |P_{\delta+1}(f,x)|>\lambda\}} u(x)\chi_{[4/R,1)}(x)\,dx\leqslant\frac{C}{\lambda}\int_0^1\left|f(x)\right|\frac{R^{\delta}u(x)}{\Phi_\nu(Rx)}\,dx,\qquad \lambda>0, \tag{19}$$

to hold with C independent of R; this follows from [1, Theorem 2] taken with $p=q=1$, $\eta=\delta+1>0$, $U(x)=R^{-\delta}u(x)\chi_{[4/R,1)}(x)$ and $V(x)=\Phi_\nu(Rx)^{-1}u(x)$ for $x\in(0,1)$ and $U(x)=V(x)=0$ for $x\geqslant 1$ and adding the supremum on R to conclude that the constant do not depend on it. The conditions (15) when $\nu\in(-1,-1/2]$ or (16) when $\nu\in(-1/2,\infty)$ are sufficient for the inequality

$$\int_{\{0<x<1:\ |Q_{-(\nu+1/2)}(f,x)|>\lambda\}} u(x)\chi_{(0,4/R)}(x)\,dx$$
$$\leqslant\frac{C}{\lambda}\int_0^1\left|f(x)\right|R^{\delta-(\nu+1/2)}x^{\delta+1}u(x)\chi_{[4/R,1)}(x)\,dx,\qquad \lambda>0, \tag{20}$$

to hold with C independent of R. In this case this follows from [1, Theorems 4 and 5] taken with $p=q=1$, $\eta=-(\nu+1/2)$, $U(x)=u(x)\chi_{(0,4/R)}(x)$ and $V(x)=R^{\delta-(\nu+1/2)}x^{\delta+1}u(x)\chi_{[4/R,1)}(x)$ for $x\in(0,1)$ and $U(x)=V(x)=0$ for $x\geqslant 1$. Again the supremum on R guarantees the uniformity of the constant C. The condition (17) is sufficient for the inequality

$$\int_{\{0\leqslant x<1:\ |Q_0(f,x)|>\lambda\}} u(x)\chi_{[4/R,1)}(x)\,dx$$

$$\leqslant \frac{C}{\lambda}\int_0^1 \left|f(x)\right|R^\delta x^{\delta+1}u(x)\chi_{[4/R,1)}(x)\,dx, \quad \lambda>0, \tag{21}$$

to hold with C independent of R; this follows from [1, Theorem 4] taken with $p=q=1$, $\eta=0$, $U(x)=u(x)\chi_{[4/R,1)}(x)$ and $V(x)=R^\delta x^{\delta+1}u(x)\chi_{[4/R,1)}(x)$ for $x\in(0,1)$ and $U(x)=V(x)=0$ for $x\geqslant 1$. In this situation, we can also make the same remarks about the supremum in R as in the previous cases. Finally, the condition (18) will be used to treatise the operator in the region A_1.

Theorem 2. *Let $\nu>-1$, $\delta>0$, and $u(x)$ be a weight on $(0,1)$ that satisfies the conditions (14), (15), (18), (5) with $p=1$, and (16) for $\nu\in(-1,-1/2]$ or (17) for $\nu\in(-1/2,\infty)$. Then*

$$\int_{\{0<x<1:\ |B_R^\delta(f,x)|>\lambda\}} u(x)\,dx \leqslant \frac{C}{\lambda}\int_0^1 \left|f(x)\right|u(x)\,dx, \quad \lambda>0, \tag{22}$$

for all $f\in L^1(u)$, with a constant C independent of R and f.

Corollary 2. *Let $\nu>-1$, $\delta>0$ and $R>0$. If*

$$-\delta-1\leqslant a\leqslant \delta \tag{23}$$

and

$$-1-(\nu+1/2)\leqslant a\leqslant \nu+1/2, \tag{24}$$

for $\nu\neq -1/2$, or

$$-1<a\leqslant 0, \tag{25}$$

for $\nu=-1/2$, then

$$\int_{\{0<x<1:\ |B_R^\delta(f,x)|>\lambda\}} x^a\,dx \leqslant \frac{C}{\lambda}\int_0^1 \left|f(x)\right|x^a\,dx, \quad \lambda>0.$$

3. Proofs of the main results

Proof of Theorem 1. Assume $u(x)$ satisfies the assumptions of the theorem. In these conditions, the operator B_R^δ is well defined for $f\in L^p(u)$ because the integrals defining the coefficients exist. Indeed, using that $|\phi_j(y)|\leqslant C\Phi_\nu(s_j y)$, we have

$$\left|\int_0^1 f(y)\phi_j(y)\,dy\right| \leqslant C\left(\int_0^1 \left(\frac{\Phi_\nu(s_j y)}{u(y)}\right)^{p'} dy\right)^{1/p'} \|f\|_{L^p(u)}$$

and the convergent of the integral is clear by using (3).

Then by using weighted Hardy's inequality (7), which follows from the condition (3), we obtain

$$\int_0^1 \left|u(x)T_3(f,x)\right|^p dx \leqslant C\int_0^1 \left|u(x)\chi_{[4/R,1)}(x)\int_0^{x/2} \frac{\Phi_\nu(Ry)}{R^\delta|x-y|^{\delta+1}} f(y)\,dy\right|^p dx$$
$$\leqslant C\int_0^1 \left(\frac{u(x)\chi_{[4/R,1)}(x)}{R^\delta x^{\delta+1}}\int_0^{x/2} \Phi_\nu(Ry)\left|f(y)\right| dy\right)^p dx$$
$$\leqslant C\int_0^1 \left|u(x)f(x)\right|^p dx.$$

Similarly, bearing in mind that (4) implies the weighted Hardy's inequality (8), we get

$$\int_0^1 \left|u(x)T_4(f,x)\right|^p dx \leqslant C\int_0^1 \left|u(x)f(x)\right|^p dx.$$

The corresponding inequality for $T_2(f,x)$ and $T_5(f,x)$ is a consequence of the estimate

$$\left|T_2(f,x)\right| + \left|T_5(f,x)\right| \leqslant CM(f,x). \tag{26}$$

We conclude the bound for these two parts with (26) and (9) (the latter following from (5)). To demonstrate (26) with T_2 it is enough to observe that

$$T_2(f,x) \leqslant CR\int_{x-\frac{4}{3R}}^{\min\{x+\frac{4}{3R},1\}} \left|f(y)\right| dy \leqslant CM(f,x).$$

To analyze (26) for the operator T_5 we use

$$A_5 = \bigcup_{k=1}^m \left(A_5 \cap \left\{(x,y)\colon\ 2^k < R|x-y| \leqslant 2^{k+1}\right\}\right),$$

with $m = [\log_2 R] - 1$. In this manner

$$T_5(f,x) \leqslant C\sum_{k=1}^m 2^{-k(\delta+1)} R \int_{\{y\colon\ 2^k<R|x-y|\leqslant 2^{k+1}\}} f(y)\,dy$$
$$\leqslant C\sum_{k=1}^m 2^{-k\delta} M(f,x) \leqslant CM(f,x).$$

Finally, for the case of $T_1(f,x)$, using Hölder's inequality, we have

$$\int_0^1 \left|u(x)T_1(f,x)\right|^p dx \leqslant C \int_0^{4/R} \left|u(x) \int_0^{4/R} (xy)^{\nu+1/2} R^{2(\nu+1)} f(y)\,dy\right|^p dx$$
$$\leqslant C R^{2p(\nu+1)} \left(\int_0^{4/R} \left(u(x)x^{\nu+1/2}\right)^p dx \right) \left(\int_0^{4/R} \left(\frac{x^{\nu+1/2}}{u(x)}\right)^{p'} dx \right)^{p/p'} \int_0^1 \left|u(x)f(x)\right|^p dx$$
$$\leqslant C \int_0^1 \left|u(x)f(x)\right|^p dx,$$

where in the last step we have used (6). □

Proof of Corollary 1. The necessity of the conditions (12) and (13) follows from the necessity part of [2, Theorem 1] taken with $A = a$. The sufficiency of (12) and (13) is a consequence of the fact that they imply conditions (3), (4) and (6). The condition (5) holds for any $a \in \mathbb{R}$. □

Proof of Theorem 2. Assume $u(x)$ satisfies the assumptions of the theorem. As in the previous theorem, the operator B_r^δ exists for $f \in L^1(u)$ because the coefficients are well defined. In this case this fact is a consequence of condition (14).

Then, by the estimate (2), we obtain

$$\int_{\{0<x<1:\ |T_3(f,x)|>\lambda\}} u(x)\,dx$$
$$\leqslant C \int_{\{0<x<1:\ x^{-(\delta+1)} \int_0^{x/2} \Phi_\nu(Ry)R^{-\delta}|f(y)|\,dy>\lambda\}} u(x)\chi_{[4/R,1)}(x)\,dx$$
$$\leqslant C \int_{\{0<x<1:\ P_{\delta+1}(\Phi_\nu(Ry)R^{-\delta}|f(y)|,x)>\lambda\}} u(x)\chi_{[4/R,1)}(x)\,dx.$$

So, using (19), which is possible by condition (14), we get

$$\int_{\{0<x<1:\ |T_3(f,x)|>\lambda\}} u(x)\,dx \leqslant \frac{C}{\lambda} \int_0^1 \left|f(x)\right|u(x)\,dx.$$

The inequality for T_4 required a thorough analysis. From the bound (2), it is obtained that

$$\int_{\{0<x<1:\ |T_4(f,x)|>\lambda\}} u(x)\,dx$$
$$\leqslant C \int_{\{0<x<1:\ x^{\nu+1/2} \int_x^1 R^{\nu+1/2-\delta}|f(y)|y^{-(\delta+1)}\,dy>\lambda\}} \chi_{(0,4/R)}(x)u(x)\,dx$$
$$+ C \int_{\{0<x<1:\ \int_x^1 R^{-\delta}y^{-(\delta+1)}|f(y)|\,dy>\lambda\}} \chi_{[4/R,1)}(x)u(x)\,dx$$

$$\leqslant C \int_{\{0<x<1:\ Q_{-(\nu+1/2)}(R^{\nu+1/2-\delta} y^{-(\delta+1)}|f(y)|,x)>\lambda\}} \chi_{(0,4/R)}(x)u(x)\,dx$$
$$+ C \int_{\{0<x<1:\ Q_0(R^{-\delta} y^{-(\delta+1)}|f(y)|,x)>\lambda\}} \chi_{[4/R,1)}(x)u(x)\,dx.$$

Now, with (15) for $\nu \in (-1,-1/2]$ and (16) for $\nu \in (-1/2,\infty)$, we can apply (20) to have

$$\int_{\{0<x<1:\ Q_{-(\nu+1/2)}(R^{\nu+1/2-\delta} y^{-(\delta+1)}|f(y)|,x)>\lambda\}} \chi_{(0,4/R)}(x)u(x)\,dx \leqslant \frac{C}{\lambda}\int_0^1 \left|f(x)\right|u(x)\,dx.$$

To complete the estimate for T_4, we consider (21), which follows from (17), to conclude that

$$\int_{\{0<x<1:\ Q_0(R^{-\delta} y^{-(\delta+1)}|f(y)|,x)>\lambda\}} \chi_{[4/R,1)}(x)u(x)\,dx \leqslant \frac{C}{\lambda}\int_0^1 \left|f(x)\right|u(x)\,dx.$$

The corresponding inequality for $T_2(f,x)$ and $T_5(f,x)$ follows from (5) as in the proof of the previous theorem by taking into account (10). Finally, for the case of $T_1(f,x)$, by using Hölder's inequality,

$$\int_0^{4/R} f(y)(xy)^{\nu+1/2} R^{2(\nu+1)}\,dy \leqslant C x^{\nu+1/2} R^{2(\nu+1)} \left\|u(y)f(y)\right\|_{L^1} \sup_{0<y<4/R} \frac{y^{\nu+1/2}}{u(y)}$$

then

$$\int_{\{0<x<1:\ |T_1(f,x)|>\lambda\}} u(x)\,dx$$
$$\leqslant C\lambda^{-1} R^{2(\nu+1)} \left\|x^{\nu+1/2}\chi_{(0,4/R)}(x)\right\|_{L^{1,\infty}(u)} \left\|u(y)f(y)\right\|_{L^1} \sup_{0<y<4/R} \frac{y^{\nu+1/2}}{u(y)}$$
$$\leqslant C\lambda^{-1}\left\|u(x)f(x)\right\|_{L^1},$$

where we have applied (18). □

Proof of Corollary 2. Clearly, (23) and (24), for $\nu \neq -1/2$, and (23) and (25), for $\nu = -1/2$, imply conditions (14)–(18). The condition (5) holds for any $a \in \mathbb{R}$. □

Acknowledgment

The authors are highly indebted to the referee for comments that improved the final version of this paper.

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