WEIGHTED TRANSPLANTATION FOR FOURIER-BESSEL SERIES

By

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Abstract. We prove weighted transplantation inequalities for Fourier–Bessel series with weights more general than previously considered power weights. These inequalities follow by using a local version of the Calderón–Zygmund operator theory. The approach also allows us to obtain weighted weak type $(1,1)$ inequalities. As a typical application of transplantation inequalities, a multiplier result for the expansions considered is proved within a weighted setting with general weights.

1 Introduction

Given $\nu > -1$, let $\lambda_{n,\nu}$, $n = 1, 2, \dots$, denote the sequence of positive zeros of the Bessel function $J_\nu(z)$. Then the functions

$$\psi_n^\nu(x) = d_{n,\nu}(\lambda_{n,\nu}x)^{1/2} J_\nu(\lambda_{n,\nu}x), \qquad d_{n,\nu} = \sqrt{2}|\lambda_{n,\nu}^{1/2} J_{\nu+1}(\lambda_{n,\nu})|^{-1},$$

$n = 1, 2, \dots$, form a complete orthonormal system in $L^2((0,1), dx)$. In particular,

$$\psi_n^{-1/2}(x) = \sqrt{2}\cos(\pi(n-1/2)x), \qquad \psi_n^{1/2}(x) = \sqrt{2}\sin(\pi n x),$$

for $n = 1, 2, \dots$. Given a function f on $(0,1)$, we associate to it its Fourier–Bessel series

$$f(x) \sim \sum_1^\infty c_n^\nu(f)\psi_n^\nu(x), \qquad c_n^\nu(f) = \int_0^1 f(x)\psi_n^\nu(x)\,dx,$$

provided that the coefficients exist. A comprehensive study of Fourier–Bessel expansions is contained in Chapter XVII of Watson's monograph [11].

*Research of the first author supported by Grant BFM2003-06335-603-03 of the DGI
†Research of the second author supported by KBN Grant # 2 P03A 028 25

The main goal of this paper is to prove a weighted transplantation theorem for Fourier–Bessel expansions. Thus, given $\mu > -1, \nu > -1$, we define the transplantation operator $T_{\mu\nu}$ on $L^2((0,1), dx)$ by the series

$$T_{\mu\nu} f(x) = \sum_{n=1}^{\infty} c_n^{\mu}(f) \psi_n^{\nu}(x), \quad f \in L^2((0,1), dx),$$

convergent in $L^2((0,1), dx)$.

In [3], we proved a transplantation result for Fourier–Bessel series by following Muckenhoupt's approach from [8]. This approach allowed us to consider power weights. In the present paper, we use the theory of Calderón–Zygmund operators to include general weights. To be precise, the transplantation operator $T_{\mu\nu}$ is a Calderón–Zygmund operator only for $\mu, \nu \geq 1/2$; see Remark 4.3. The notion of a local Calderón–Zygmund operator, introduced in [10] by Nowak and Stempak (and implicitly included in [8]), is applicable also in our case, allowing us to treat the whole range of parameters $\mu, \nu > -1$. Even in the case $\mu, \nu \geq 1/2$, treating $T_{\mu\nu}$ as a local Calderón–Zygmund operator brings an additional advantage: more weights are admitted when comparing with results that could be obtained by using the (global) CZ theory. The reason for this is that for $\nu, \mu \geq 1/2$, the condition (2.8) is stronger than any of the conditions (2.1) and (2.2), or (2.11) and (2.13) (also (2.3) is weaker than (2.8)); see Proposition 2.4.

In addition, our new approach also gives weighted weak type $(1,1)$ inequalities (these were not discussed in [3]). Since Fourier–Bessel expansions should be treated as discrete analogues of Hankel transforms, it is worth noting that the approach we use here is a natural counterpart to the approach developed in [10] for the Hankel transform. Let us also mention that in [3], we considered a slightly more general setting by allowing a shift in the order parameter, i.e., by writing ψ_{n+m}^{ν}, m an integer, in the definition of $T_{\mu\nu} f$. Here, for the sake of simplicity, we omit this generalization (but our main results remain valid in this more general setting as well).

Denoting by $L(x,y)$ the kernel associated with the operator $T_{\mu\nu}$, we consider the decomposition

$$T_{\mu\nu} = T_{\mu\nu}^1 + T_{\mu\nu}^2 + T_{\mu\nu}^3,$$

where $T_{\mu\nu}^1$ and $T_{\mu\nu}^2$ are integral operators related to the kernels

$$L(x,y)\chi_{\{(x,y):0<y<x/2\}} \quad \text{and} \quad L(x,y)\chi_{\{(x,y):\min\{1,3x/2\}<y<1\}},$$

respectively, and $T_{\mu\nu}^3 = T_{\mu\nu} - T_{\mu\nu}^1 - T_{\mu\nu}^2$. The operators $T_{\mu\nu}^1$ and $T_{\mu\nu}^2$ will be analyzed by means of weighted Hardy inequalities. To handle $T_{\mu\nu}^3$, we apply the aforementioned local Calderón–Zygmund theory.

Throughout the paper, we use fairly standard notation. Thus, for a weight w on $(0,1)$ (a nonnegative measurable function such that $w(x)<\infty$, a.e.), $L^p(w)$ and $L^{1,\infty}(w)$ denote the weighted L^p and the weighted weak L^1 spaces (with respect to Lebesgue measure) consisting of all functions f on $(0,1)$ for which

$$\|f\|_{L^p(w)} = \Big(\int_0^1 |f(x)w(x)|^p\,dx\Big)^{1/p} < \infty$$

or

$$\|f\|_{L^{1,\infty}(w)} = \sup_{t>0}\Big(t\int_{\{0<x<1:|f(x)|>t\}} w(x)\,dx\Big) < \infty,$$

respectively. If $w\equiv 1$, we simply write L^p or $L^{1,\infty}$. For $1\le p\le\infty$, p' denotes its conjugate, $1/p+1/p'=1$. By $\langle f,g\rangle$ we mean $\int_0^1 f(x)\overline{g(x)}\,dx$ whenever the integral makes sense. We frequently write CZ to abbreviate the term "Calderón-Zygmund".

The structure of the paper is as follows. In Section 2, we state the main results of the paper; they are contained in Theorems 2.1, 2.2 and 2.3. Section 3 is focused in the development of the theory of local CZ operators in our setting. In Section 4, we define and analyze the kernel of the transplantation operator $T_{\mu\nu}$. To prove the main results of this section, Proposition 4.1 and Proposition 4.2, we heavily exploit arguments from our previous paper [3]. Finally, in Section 5, we provide proofs of the main results.

The authors are highly indebted to the referee for comments that improved the presentation.

2 Statement of results

Given a weight function w on $(0,1)$, consider the following set of conditions:

$$\sup_{0<r<1}\Big(\int_r^1 w(x)^p x^{-p(\mu+3/2)}\,dx\Big)^{1/p}\Big(\int_0^r w(x)^{-p'}x^{p'(\mu+1/2)}\,dx\Big)^{1/p'}<\infty, \tag{2.1}$$

$$\sup_{0<r<1}\Big(\int_0^r w(x)^p x^{p(\nu+1/2)}\,dx\Big)^{1/p}\Big(\int_r^1 w(x)^{-p'}x^{-p'(\nu+3/2)}\,dx\Big)^{1/p'}<\infty, \tag{2.2}$$

$$\sup_{0<u<v<\min\{1,2u\}}\frac{1}{v-u}\Big(\int_u^v w(x)^p\,dx\Big)^{1/p}\Big(\int_u^v w(x)^{-p'}\,dx\Big)^{1/p'}<\infty. \tag{2.3}$$

For a weight w satisfying (2.3), we write $w^p\in A_{p,\mathrm{loc}}(0,1)$ and say that w^p is a local A_p weight. The left side of (2.3) is then called the $A_{p,\mathrm{loc}}$ norm of w^p. We admit $1\le p<\infty$ when considering conditions (2.1), (2.2), and (2.3). Here and later on, for $p' = \infty$ integrals of the form appearing in (2.1)–(2.3) have the usual interpretation. For example, the second factor in (2.1) is taken as $\operatorname{ess\,sup}_{x\in(0,r)}[w(x)^{-1}x^{\mu+1/2}]$. It is easily seen that for a power weight function $w(x) = x^a$, $a \in \mathbb{R}$, (2.1) is satisfied if and only if $a < -1/p+(\mu+3/2)$, (2.2) is satisfied if and only if $a > -(\nu+1/2)-1/p$, and (2.3) is satisfied for each $a \in \mathbb{R}$. The condition (2.1) is necessary and sufficient for the weighted Hardy inequality

(2.4)
$$\Big(\int_0^1 \Big|w(x)x^{-(\mu+3/2)}\int_0^x f(t)\,dt\Big|^p\,dx\Big)^{1/p} \le C\Big(\int_0^1 \Big|w(x)x^{-(\mu+1/2)}f(x)\Big|^p\,dx\Big)^{1/p}$$

to hold, while the condition (2.2) is necessary and sufficient for its dual version

$$\Big(\int_0^1 \Big|w(x)x^{\nu+1/2}\int_x^1 f(t)\,dt\Big|^p\,dx\Big)^{1/p} \le C\Big(\int_0^1 \Big|w(x)x^{\nu+3/2}f(x)\Big|^p\,dx\Big)^{1/p} \tag{2.5}$$

to be satisfied; this follows from [7, Theorems 1 and 2]. For $1 < p < \infty$, the local A_p condition (2.3) for w^p is sufficient for the estimate

$$\int_0^1 |Tf(x)w(x)|^p\,dx \le C\int_0^1 |f(x)w(x)|^p\,dx \tag{2.6}$$

to hold, where T represents a local Calderón–Zygmund operator (see Definition 3.2 below). When $p = 1$, (2.3) is sufficient for the weighted weak type (1,1) inequality

$$\int_{\{0<x<1:|Tf(x)|>\lambda\}} w(x)\,dx \le \frac{C}{\lambda}\int_0^1 |f(x)|w(x)dx, \qquad \lambda > 0, \tag{2.7}$$

to hold. These estimates for local Calderón–Zygmund operators are contained in Theorem 3.2 (see also [10, Section 4]).

Recall that the (global) A_p condition for w^p, $1 \le p < \infty$, is

$$\sup_{0\le u<v\le 1}\frac{1}{v-u}\left(\int_u^v w(x)^p\,dx\right)^{1/p}\left(\int_u^v w(x)^{-p'}\,dx\right)^{1/p'} < \infty. \tag{2.8}$$

Here, as in (2.3), the second integral is understood as $\operatorname{ess\,sup}_{(u,v)} w^{-1}$. If a weight w satisfies (2.8), we write $w^p \in A_p(0,1)$ and denote the left side of (2.8) by $\|w^p\|_{A_p}$. Finally, note that if a weight w on $(0,1)$ satisfies any of the conditions (2.1)–(2.3) or (2.8), then either $w \equiv 0$ or $w(x) > 0$ x-a.e. (here the convention $0\cdot\infty = 0$ is used); and the same remark applies to the conditions (2.11)–(2.13).

The main results of the paper are contained in the following theorems.

Theorem 2.1. *Let $\mu,\nu > -1$, $\mu \neq \nu$, and $1 < p < \infty$. Let the weight w satisfy conditions* (2.1)*,* (2.2) *and* (2.3)*. Then*

$$\left(\int_0^1 \left|T_{\mu\nu} f(x) w(x)\right|^p dx\right)^{1/p} \leq C\left(\int_0^1 |f(x)w(x)|^p dx\right)^{1/p} \tag{2.9}$$

for all $f \in L^2 \cap L^p(w)$. Consequently, $T_{\mu\nu}$ extends uniquely to a bounded linear operator on $L^p(w)$, and denoting this extension again by $T_{\mu\nu}$, we have for $f \in L^p(w)$,

$$T_{\mu\nu} f(x) \sim \sum_{n=1}^{\infty} c_n^{\mu}(f)\psi_n^{\nu}(x). \tag{2.10}$$

In order to treat weighted weak type (1,1) inequalities for the transplantation operator for a given weight function w on (0,1), consider the following set of conditions:

$$\sup_{0<r<1}\left(\int_r^1 \left(\frac{r}{x}\right)^{\delta} \frac{w(x)}{x^{\mu+3/2}}\,dx\right)\left(\operatorname*{ess\,sup}_{x\in(0,r)} \frac{x^{\mu+1/2}}{w(x)}\right) < \infty, \tag{2.11}$$

$$\sup_{0<r<1} r^{\nu+1/2}\left(\int_0^r w(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in(r,1)} \frac{1}{x^{\nu+3/2}w(x)}\right) < \infty, \tag{2.12}$$

$$\sup_{0<r<1}\left(\int_0^r \left(\frac{x}{r}\right)^{\delta} x^{\nu+1/2} w(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in(r,1)} \frac{1}{x^{\nu+3/2}w(x)}\right) < \infty. \tag{2.13}$$

In (2.11) and (2.13), we assume that there exists $\delta > 0$ such that the corresponding quantities are finite. Moreover, (2.12) is considered for $\nu \in (-1,-1/2]$, while (2.13) is taken into account for $\nu \in (-1/2,\infty)$. It is easily seen that for a power weight function $w(x) = x^a$, $a \in \mathbb{R}$, (2.11) is satisfied if and only if $a \leq \mu + 1/2$ and (2.12) and (2.13) are satisfied if and only if $a \geq -(\nu+3/2)$ ($>$ if $\nu = -1/2$). Let P_η, Q_η, η real, denote the Hardy operators acting on functions defined on $(0,1)$:

$$P_\eta f(x) = x^{-\eta}\int_0^x f(t)\,dt, \quad Q_\eta f(x) = x^{-\eta}\int_x^1 f(t)\,dt, \quad 0 < x < 1.$$

The condition (2.11) is necessary and sufficient for the inequality

$$\int_{\{0<x<1:|P_{\mu+3/2}f(x)|>\lambda\}} w(x)\,dx \leq \frac{C}{\lambda}\int_0^1 |f(x)|x^{-(\mu+1/2)}w(x)dx, \quad \lambda > 0, \tag{2.14}$$

to hold; this follows from [2, Theorem 2] with $p = q = 1$, $\eta = \mu + 3/2 > 0$, $U(x) = w(x)$ and $V(x) = x^{-(\mu+1/2)}w(x)$ for $x \in (0,1)$ and $U(x) = V(x) = 0$ for

$x \geq 1$. The condition (2.12) in the case $\nu \in (-1, -1/2]$ and the condition (2.13) in the case $\nu \in (-1/2, \infty)$ are necessary and sufficient for the inequality

$$\int_{\{0<x<1: |Q_{-(\nu+1/2)} f(x)| > \lambda\}} w(x)\, dx \leq \frac{C}{\lambda} \int_0^1 |f(x)| x^{\nu+3/2} w(x) dx, \quad \lambda > 0, \tag{2.15}$$

to hold; this follows from [2, Theorem 4] and [2, Theorem 5] with $p = q = 1$, $\eta = -(\nu + 1/2)$, $U(x) = w(x)$ and $V(x) = x^{\nu+3/2} w(x)$ for $x \in (0, 1)$ and $U(x) = V(x) = 0$ for $x \geq 1$.

Theorem 2.2. *Let $\mu, \nu > -1$, $\mu \neq \nu$, and let w be a weight on $(0,1)$ that satisfies* (2.11)*,* (2.3) *with $p = 1$, and either* (2.12) *or* (2.13)*, depending on whether $\nu \in (-1, -1/2]$ or $\nu \in (-1/2, \infty)$. Then*

$$\int_{\{0<x<1: |T_{\mu\nu} f(x)| > \lambda\}} w(x)\, dx \leq \frac{C}{\lambda} \int_0^1 |f(x)| w(x)\, dx, \quad \lambda > 0,$$

for all $f \in L^2 \cap L^1(w)$. Consequently, $T_{\mu\nu}$ extends uniquely to a bounded linear operator from $L^1(w)$ to $L^{1,\infty}(w)$.

As usual, transplantation theorems are used in proving multiplier theorems. A bounded sequence $\{m_n\}_{n=1}^{\infty}$ is called an $L^p(w)$ multiplier for the $\{\psi_n^\nu\}$-expansions if

$$\Big\| \sum_{n=1}^{\infty} m_n d_n \psi_n^\nu \Big\|_{L^p(w)} \leq D \Big\| \sum_{n=1}^{\infty} d_n \psi_n^\nu \Big\|_{L^p(w)},$$

with $D > 0$ independent of any sequence $d = \{d_n\}_{n=1}^{\infty}$ such that $d_n = 0$ for all but finitely many indices. We say that a bounded sequence $\{m_n\}_{n=1}^{\infty}$ satisfies Marcinkiewicz's condition provided

$$B(m, 1, 1) = \sup_{k \geq 1} \sum_{n=2^{k-1}}^{2^k - 1} |m_n - m_{n+1}| < \infty.$$

The class of all sequences satisfying the above condition is denoted by $M(1, 1)$.

Theorem 2.3. *Let $\nu \geq -1/2$, $1 < p < \infty$, and let w be a weight on $(0, 1)$ satisfying* (2.1) *with μ replaced by ν and* (2.2)*. In addition, assume that $w^p \in A_p(0, 1)$ and $m = \{m_n\}_{n=1}^{\infty} \in M(1, 1)$. Then*

$$\Big\| \sum_{n=1}^{\infty} m_n d_n \psi_n^\nu \Big\|_{L^p(w)} \leq C B(m, 1, 1) \Big\| \sum_{n=1}^{\infty} d_n \psi_n^\nu \Big\|_{L^p(w)}, \tag{2.16}$$

with $C > 0$ not depending on m and $d = \{d_n\}_{n=1}^{\infty}$, where $d_n = 0$ for all but finitely many indices. Consequently, the linear operator $\psi_n^\nu \mapsto d_n \psi_n^\nu$, initially defined on the subspace spanned by $\{\psi_n^\nu : n = 1, 2, \dots\}$, extends uniquely to a bounded operator on $L^p(w)$.

More subtle versions of the above theorem could be stated and proved; however, this would require a more complicated set of assumptions.

Finally, we state two result which clarify relations between different conditions we imposed on w.

Proposition 2.4. *Let $\mu,\nu \ge -1/2$, $1 \le p < \infty$, and let w be a weight satisfying* (2.8)*. Then for $1 < p < \infty$, w verifies* (2.1) *and* (2.2)*, while in the case $p = 1$, w satisfies* (2.11) *together with* (2.13) *when $\nu > -1/2$ or* (2.12) *when $\nu = -1/2$.*

Proposition 2.5. *Let $1 < p < \infty$ be given. The class of weights w that satisfy* (2.1) *with $\mu = -1/2$,* (2.2) *with $\nu = -1/2$ and* (2.3) *(so that w^p is a local A_p weight) coincides with the class of weights w that satisfy* (2.8) *(so that w^p is an A_p weight).*

3 Local CZ kernels

It is clear that the CZ theory (specified to $\mathbb{R}$) works, with appropriate adjustments, when the underlying space is $(0,1)$ equipped with the Lebesgue measure dx. Thus we use properly adjusted facts from the classic CZ theory (presented, for instance, in [4]) to the aforementioned setting without further comments. In addition, we follow rather closely [10, Section 4], where the notion of local CZ operator on $(0,\infty)\times(0,\infty)$ was introduced. In particular, we prove Proposition 3.1 by adapting arguments used in the proof of [10, Proposition 4.1].

Let $\Delta = \{(x,x) : x \in (0,1)\}$ be the diagonal of $(0,1)\times(0,1)$. We say (cf. [4, p. 99]) that $K\colon (0,1)\times(0,1)\setminus\Delta \longrightarrow \mathbb{C}$ is a standard kernel on $(0,1)\times(0,1)$ if, for $x,y,z \in (0,1)$,

$$|K(x,y)| \le C|x-y|^{-1}, \tag{3.1}$$

$$|K(x,y)-K(x,z)| \le C|y-z||x-y|^{-2} \quad \text{if } |x-y| > 2|y-z|, \tag{3.2}$$

$$|K(x,y)-K(z,y)| \le C|x-z||x-y|^{-2} \quad \text{if } |x-y| > 2|x-z|. \tag{3.3}$$

Note that by (3.2) and (3.3), standard kernels are continuous. Clearly, they also satisfy the Hörmander conditions

$$\int_{\{x\in(0,1):|x-y|>2|y-z|\}} |K(x,y)-K(x,z)|\,dx \le C,$$

$$\int_{\{y\in(0,1):|x-y|>2|x-w|\}} |K(x,y)-K(w,y)|\,dy \le C,$$

for all $x,y,w,z \in (0,1)$. It is also clear that the gradient condition

$$|\nabla K(x,y)| \le C|x-y|^{-2}, \tag{3.4}$$

implies (3.2) and (3.3).

Definition 3.1. A **local standard kernel** on $(0,1)\times(0,1)$ is a kernel $K\colon(0,1)\times(0,1)\backslash\Delta\to\mathbb{C}$ supported in the region

$$\mathcal{D}=\{(x,y)\in(0,1)\times(0,1):0<x/2\le y\le 3x/2\}$$

and satisfying (3.1)–(3.3) on $\mathcal{D}$.

Definition 3.2. An operator T is a local Calderón–Zygmund operator if

1) T is bounded on L^2;

2) there exists a local standard kernel K associated with T such that

$$\langle Tf,g\rangle=\int_0^1\int_{x/2}^{\min\{1,3x/2\}}K(x,y)f(y)\overline{g(x)}\,dy\,dx$$

for all $f,g\in C_c^\infty(0,1)$ with disjoint supports.

Proposition 3.1. *Let $K(x,y)$ be a local standard kernel on $(0,1)\times(0,1)$. Then K satisfies the Hörmander type conditions*

$$\int_{(0,1)\backslash 2I}|K(x,y)-K(x,z)|\,|f(x)|\,dx\le CMf(y),\quad y,z\in I,\tag{3.5}$$

$$\int_{(0,1)\backslash 2I}|K(x,y)-K(w,y)|\,|f(y)|\,dy\le CMf(x),\quad x,w\in I,\tag{3.6}$$

for all intervals $I\subset(0,1)$. Here M denotes the (non-centered) Hardy-Littlewood maximal function on $(0,1)$

$$Mf(x)=\sup_{0\le u<x<v\le 1}\frac{1}{v-u}\int_u^v|f(y)|\,dy,$$

and $2I$ is the interval with the same center as I and such that $|2I|=2|I|$.

Proof. We focus on proving (3.5), since the proof of (3.6) is entirely analogous. Let $I=(u,v)\subset(0,1)$. We assume that $u<y<z<v$; the analysis of the case $z<y$ is similar.

Since $2I=((3u-v)/2,(3v-u)/2)$, by the assumption on the support of K, the region of integration in (3.5), is the set

$$(y/2,\min\{3z/2,1\})\backslash\big((3u-v)/2,(3v-u)/2\big).$$

Note, that the supports of $K(\cdot,y)$ and $K(\cdot,z)$ overlap only on $(z/2,\min\{3y/2,1\})$.

Thus, proving (3.5) reduces to showing that each of the three integrals

$$I_1 = \int_{B_1} |K(x,y)|\,|f(x)|\,dx, \quad B_1 = (y/2, \min\{z/2, (3u-v)/2\}),$$

$$I_2 = \int_{B_2} |K(x,y) - K(x,z)|\,|f(x)|\,dx,$$

$$B_2 = (z/2, (3u-v)/2) \cup ((3v-u)/2, \min\{3y/2, 1\}),$$

$$I_3 = \int_{B_3} |K(x,z)|\,|f(x)|\,dx, \quad B_3 = (\max\{3y/2, (3v-u)/2\}, \min\{3z/2, 1\}),$$

is bounded by the right side of (3.5). Here we use the convention that $(a,b) = \emptyset$ if $a \geq b$.

Consider I_1 first. If $v/u \leq 3/2$, then $B_1 = (y/2, z/2)$; and for $x \in (y/2, z/2)$, we have $y - x > y - z/2 > y/4$ (the last inequality follows since $z < v \leq 3u/2 < 3y/2$). Thus

$$I_1 \leq C \int_{y/2}^{z/2} \frac{|f(x)|}{|y-x|}\,dx \leq \frac{C}{y} \int_{y/2}^{z/2} |f(x)|\,dx \leq \frac{C}{y} \int_{y/2}^{\min\{3y/2,1\}} |f(x)|\,dx \leq CMf(y).$$

If $v/u > 3/2$, then $3/4 > (3 - v/u)/2$. Hence, for $x \in (y/2, \min\{z/2, (3u-v)/2\})$, $y - x > y/4$. This is because $3y/4 > 3u/4 > (3u-v)/2$; therefore,

$$y - x > y - (3u-v)/2 > y/4.$$

Consequently,

$$I_1 \leq C \int_{y/2}^{(3u-v)/2} \frac{|f(x)|}{|y-x|}\,dx \leq \frac{C}{y} \int_{y/2}^{\min\{3y/2,1\}} |f(x)|\,dx \leq CMf(y).$$

Considering I_2, we write $\ell = v - u$ and use conditions (3.1), (3.2) to get

$$I_2 \leq C \int_{B_2} \frac{|y-z|}{|x-y|^2}\,|f(x)|\,dx \leq C\ell \int_{B_2} \frac{|f(x)|}{|x-y|^2}\,dx.$$

The last integral multiplied by ℓ is less than (in fact, the series below terminates)

$$\begin{aligned}
&\ell \sum_{k=-1}^{\infty} \int_{\{x\in(0,1):2^k\ell<|x-y|<2^{k+1}\ell\}} \frac{|f(x)|}{|x-y|^2}\chi_{(z/2,\min\{3y/2,1\})}(x)\,dx \\
&\quad \leq \ell \sum_{k=-1}^{\infty} \frac{1}{(2^k\ell)^2} \int_{\{x\in(0,1):|x-y|<2^{k+1}\ell\}} |f(x)|\chi_{(z/2,\min\{3y/2,1\})}(x)\,dx \\
&\quad \leq \sum_{k=-1}^{\infty} \frac{1}{2^k}\frac{1}{2^k\ell} \int_{\{x\in(0,1):|x-y|<2^{k+1}\ell\}\cap(z/2,\min\{3y/2,1\})} |f(x)|\,dx \\
&\quad \leq 4\Big(\sum_{k=-1}^{\infty} 2^{-k}\Big) Mf(y).
\end{aligned}$$

Finally, consider I_3. If $v/u \le 4/3$, then $B_3 = (3y/2, \min\{3z/2, 1\})$; and for $x \in (3y/2, \min\{3z/2, 1\})$, we have $x - z > 3y/2 - z > y/6$ (the last inequality follows from the fact that $z < v \le 4u/3 < 4y/3$). Thus

$$I_3 \le C \int_{3y/2}^{\min\{3z/2,1\}} \frac{|f(x)|}{|x-z|}\, dx \le \frac{C}{y} \int_{y/2}^{\min\{2y,1\}} |f(x)|\, dx \le CMf(y).$$

If $v/u > 4/3$, then $(3 - u/v)/2 > 9/8$. Hence, for $x \in (\max\{3y/2, (3v-u)/2\}, \min\{3z/2, 1\})$, $x - z > z/8$. This is because $9z/8 < 9v/8 < (3v-u)/2$; therefore, $x - z > (3v-u)/2 - z > z/8$. Accordingly,

$$I_3 \le C \int_{3y/2}^{\min\{3z/2,1\}} \frac{|f(x)|}{|x-z|}\, dx \le \frac{C}{z} \int_{0}^{\min\{3z/2,1\}} |f(x)|\, dx \le CMf(y). \qquad \square$$

Consequences of the estimates from Proposition 3.1 are exactly like those formulated in [10, Proposition 4.2] and [10, Theorem 4.3] for local CZ operators in $L^2(0,\infty)$. Here we state a counterpart of [10, Theorem 4.3] for the sake of completeness; we do not provide the proof, since it is a straightforward modification of the corresponding proof from [10]. The result of Theorem 3.2 is essential in the proofs of our main results.

Theorem 3.2. *Assume that T is a local Calderón–Zygmund operator and let w be a weight on $(0,1)$ such that $w^p \in A_{p,\mathrm{loc}}(0,1)$.*

(a) *If $1 < p < \infty$, then T extends to a bounded linear operator on $L^p(w)$;*

(b) *if $p = 1$, then T extends to a bounded linear operator from $L^1(w)$ to $L^{1,\infty}(w)$.*

Moreover, the corresponding L^p and weak type constants depend on w only through the $A_{p,\mathrm{loc}}$ norm of w^p.

4 The kernel of $T_{\mu\nu}$

We start with obtaining estimates of the transplantation kernel

$$L(r,x,y) = \sum_{n=1}^{\infty} r^n \psi_n^\nu(x)\psi_n^\mu(y), \tag{4.1}$$

which is the integral kernel of the operator

$$T_{\mu\nu,r} f(x) = \sum_{n=1}^{\infty} r^n c_n^\mu(f)\psi_n^\nu(x),$$

i.e.,

$$T_{\mu\nu,r} f(x) = \int_0^1 L(r,x,y) f(y)\, dy.$$

Proposition 4.1. *Let* $\mu, \nu > -1$. *Then*

$$|L(r,x,y)| \le C \begin{cases} x^{-\mu-3/2}y^{\mu+1/2}, & 0 < y \le x/2, \\ |x-y|^{-1}, & x/2 < y < \min\{1, 3x/2\}, \\ x^{\nu+1/2}y^{-\nu-3/2}, & \min\{1,3x/2\} \le y < 1, \end{cases} \tag{4.2}$$

and

$$|\nabla L(r,x,y)| \le C|x-y|^{-2}, \quad x/2 < y < \min\{1, 3x/2\}. \tag{4.3}$$

In both cases, C *is independent of* $0 < r < 1$, $0 < x < 1$ *and* $0 < y < 1$.

Proof. The first and third estimate in (4.2) is included in [3, Proposition 4.3]; the middle estimate in (4.2) follows from (5.1) in [3, Proposition 5.1] (and the estimates contained in [3, Lemma 3.1] for $n = 0$). Therefore, we concentrate on proving (4.3).

To show (4.3), it is enough to verify that

$$\left|\frac{\partial}{\partial y} L(r,x,y)\right| \le \frac{C}{(x-y)^2},$$

since the result for $\frac{\partial}{\partial x}$ follows in a similar fashion. By using the identity

$$J'_\mu(z) = (\mu/z)J_\mu(z) - J_{\mu+1}(z),$$

we have

$$\frac{d\psi_n^\mu(y)}{dy} = \frac{2\mu+1}{2y}\psi_n^\mu(y) - d_{n,\mu}\lambda_{n,\mu}^{3/2} y^{1/2} J_{\mu+1}(\lambda_{n,\mu} y).$$

In this way,

$$\frac{\partial L}{\partial y}(r,x,y) = \frac{2\mu+1}{y}L(r,x,y) - \sum_{n=1}^{\infty} r^n \psi_n^\nu(x) d_{n,\mu}\lambda_{n,\mu}^{3/2} y^{1/2} J_{\mu+1}(\lambda_{n,\mu} y). \tag{4.4}$$

For the first summand on the right of (4.4), is clear from (4.2) that

$$\left|\frac{2\mu+1}{y}L(r,x,y)\right| \le \frac{C}{y|x-y|} \le \frac{C}{(x-y)^2}.$$

To treat the second summand, we write

$$\widetilde{L}(r,x,y) = \sum_{n=1}^{\infty} r^n \psi_n^\nu(x) d_{n,\mu}\lambda_{n,\mu}(\lambda_{n,\mu} y)^{1/2} J_{\mu+1}(\lambda_{n,\mu} y)$$

and modify the argument from the proof of [3, Proposition 5.1], case $s = 0$, proving that

$$|\widetilde{L}(r,x,y)| \le C|x-y|^{-2}, \quad x/2 < y < \min\{1, 3x/2\}. \tag{4.5}$$

We use (see [5, p. 122])

$$\sqrt{z}J_\nu(z)=\sum_{j=0}^{M}\left(\frac{A_{\nu,j}}{z^j}\sin z+\frac{B_{\nu,j}}{z^j}\cos z\right)+H_M(z),\quad z\to\infty, \tag{4.6}$$

with $M=2$, where $|H_M(z)|\le Cz^{-(M+1)}$, to expand $(\lambda_{n,\nu}x)^{1/2}J_\nu(\lambda_{n,\nu}x)$ as well as $(\lambda_{n,\mu}y)^{1/2}J_{\mu+1}(\lambda_{n,\mu}y)$. Then, taking $N=\left[\frac{1}{x}\right]\sim\left[\frac{1}{y}\right]$, we write

$$\widetilde{L}(r,x,y)=F(x,y)+\sum_{j,l=0}^{2}x^{-j}y^{-l}O_{j,l}(x,y)+J_1(x,y)+J_2(x,y)+G(x,y).$$

Here

$$F(x,y)=\sum_{n=1}^{N-1}r^n\psi_n^\nu(x)d_{n,\mu}\lambda_{n,\mu}(\lambda_{n,\mu}y)^{1/2}J_{\mu+1}(\lambda_{n,\mu}y);$$

for the remainder sum that starts from $n=N$, the $O_{j,l}$ terms capture the part coming from the main parts of the expansions and are the sums of four terms of the form

$$D_{j,l}\sum_{n=N}^{\infty}r^n d_{n,\nu}d_{n,\mu}\lambda_{n,\nu}^{-j}\lambda_{n,\mu}^{-l+1}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\nu}x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\mu}y),$$

($D_{j,l}$ is a product of $A_{\nu,j}$ or $B_{\nu,j}$ and $A_{\mu+1,l}$ or $B_{\mu+1,l}$ depending on the choice of the sine or cosine); J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one, hence its absolute value is bounded by

$$\begin{aligned}|J_1(x,y)|\le&C\left|\sum_1^2\sum_{n=N}^{\infty}r^n d_{n,\nu}d_{n,\mu}\lambda_{n,\mu}H_2(\lambda_{n,\nu}x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\mu}y)\right|\\&+Cy^{-1}\left|\sum_1^2\sum_{n=N}^{\infty}r^n d_{n,\nu}d_{n,\mu}H_2(\lambda_{n,\nu}x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\mu}y)\right|\\&+Cy^{-2}\left|\sum_1^2\sum_{n=N}^{\infty}r^n d_{n,\nu}d_{n,\mu}\lambda_{n,\mu}^{-1}H_2(\lambda_{n,\nu}x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\lambda_{n,\mu}y)\right|\end{aligned}$$

(the sign $\sum_1^2$ indicates that we add two series, one for the choice of the sine, the other for the cosine); J_2 acts as J_1 but with the position of the both expansions switched, and its absolute value is controlled by

$$\begin{aligned}|J_2(x,y)| \leq & C\bigg|\sum_1^2 \sum_{n=N}^{\infty} r^n d_{n,\nu} d_{n,\mu} \lambda_{n,\mu} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\lambda_{n,\nu} x) H_2(\lambda_{n,\mu} y)\bigg| \\ &+ Cx^{-1}\bigg|\sum_1^2 \sum_{n=N}^{\infty} r^n d_{n,\nu} d_{n,\mu} \lambda_{n,\mu} \lambda_{n,\nu}^{-1} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\lambda_{n,\nu} x) H_2(\lambda_{n,\mu} y)\bigg| \\ &+ Cx^{-2}\bigg|\sum_1^2 \sum_{n=N}^{\infty} r^n d_{n,\nu} d_{n,\mu} \lambda_{n,\mu} \lambda_{n,\nu}^{-2} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\lambda_{n,\nu} x) H_2(\lambda_{n,\mu} y)\bigg|;\end{aligned}$$

finally, G captures the part that comes from the remainders,

$$G(x,y) = \sum_{n=N}^{\infty} r^n d_{n,\nu} d_{n,\mu} \lambda_{n,\mu} H_2(\lambda_{n,\nu} x) H_2(\lambda_{n,\mu} y).$$

We now analyze separately each of the thirteen summands in the above decomposition of $\widetilde{L}$ and bound them by $C|x-y|^{-2}$. In what follows, we frequently use the fact that $d_{n,\mu} = O(1)$ and $\lambda_{n,\mu} = O(n)$, and similarly for $d_{n,\nu}$ and $\lambda_{n,\nu}$.

For $F(x,y)$, using [3, (2.1), (2.2), (2.5)], we have

$$|F(x,y)| \leq Cx^{\nu+1/2} y^{\mu+3/2} \sum_{n=1}^{N-1} n^{\nu+\mu+3} \leq Cx^{\nu+\mu+2} N^{\nu+\mu+4} \leq Cx^{-2},$$

which is dominated by $C|x-y|^{-2}$ in the region considered.

For $J_1(x,y)$ (the same reasoning works for $J_2(x,y)$), using $H_2(z) = O(z^{-3})$, $z \geq 1$, and again [3, (2.1), (2.2)], we show that

$$|J_1(x,y)| \leq Cx^{-3}\Big(\sum_{n=N}^{\infty} n^{-2} + y^{-1} \sum_{n=N}^{\infty} n^{-3} + y^{-2} \sum_{n=N}^{\infty} n^{-4}\Big) \leq Cx^{-2}.$$

In a similar way, we show that

$$|G(x,y)| \leq C(xy)^{-3} \sum_{n=N}^{\infty} n^{-5} \leq Cx^{-6} N^{-4} \leq Cx^{-2}.$$

The remainder part of the proof is concerned with a more delicate analysis of the $x^{-j} y^{-l} O_{j,l}(x,y)$ terms. We start with the $x^{-2} y^{-2} O_{2,2}(x,y)$ term. It is clear that

$$|x^{-2} y^{-2} O_{2,2}(x,y)| \leq Cx^{-4} \sum_{n=N}^{\infty} n^{-3} \leq Cx^{-4} N^{-2} \leq Cx^{-2}.$$

Similarly, the same bound is obtained for $|x^{-2} y^{-1} O_{2,1}(x,y)|$ and $|x^{-1} y^{-2} O_{1,2}(x,y)|$.

Using [3, Lemma 4.1] (it is immediate to check that $\cos(\lambda_{n,\mu} y)$ may be replaced by $\sin(\lambda_{n,\mu} y)$ in this lemma; we use this extended form of [3, Lemma 4.1] here and later on) and [3, Lemma 4.2] with $j = 2$, $\gamma = 1$ and $\ell = 0$, the estimate $|x^{-2}O_{2,0}(x,y)| \le C|x-y|^{-2}$ follows once we show that

$$\frac{1}{x^2}\Bigg|\sum_{n=N}^{\infty}\frac{r^n}{n}E_{2,1,0}(n,x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi nx)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi ny)\Bigg| \le C\frac{1}{x^2}\log\left(\frac{2x}{|x-y|}\right),$$

where $E_{2,1,0}(n,x,y)$ is as in [3, (4.3)]. The form of $E_{2,1,0}$ reduces this task to showing the estimates

$$\Bigg|\sum_{n=N}^{\infty}\frac{r^n}{n}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi nx)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi ny)\Bigg| \le C\log\left(\frac{2x}{|x-y|}\right) \tag{4.7}$$

and

$$\Bigg|\sum_{n=N}^{\infty}\frac{r^n}{n}q_n(x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi nx)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi ny)\Bigg| \le C,$$

where $|q_n(x,y)| \le Cn^{-1}$. This last series is absolutely convergent, and the bound follows. The estimate (4.7) is the same as [3, (5.3)] and was proved there. The estimate for $y^{-2}O_{0,2}(x,y)$ follows in a similar fashion ($E_{0,-1,0}$ is involved).

Estimating $x^{-1}y^{-1}O_{1,1}(x,y)$, we use [3, Lemma 4.1] and [3, Lemma 4.2] with $j = 1$, $\gamma = 0$ and $\ell = 0$, and end up with the situation of the two previous estimates.

Estimating $x^{-1}O_{1,0}(x,y)$, we use [3, Lemma 4.1] and [3, Lemma 4.2] with $j = 1$, $\gamma = 1$ and $\ell = 1$ to show that $x^{-1}O_{1,0}(x,y)$ is the sum of four terms of the form

$$x^{-1}u(x)v(y)\sum_{n=N}^{\infty}r^nE_{1,1,1}(n,x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi nx)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi ny),$$

where u and v are bounded functions on $(0,1)$, or, after applying trigonometric identities and expanding $E_{1,1,1}(n,x,y)$, the sum of four terms of the form

$$x^{-1}\tilde{u}(x)\tilde{v}(y)\sum_{n=N}^{\infty}r^n\Big(A_0+\frac{A_1(x,y)}{n}+q_n^{(1)}(x,y)\Big)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y)); \tag{4.8}$$

here $\tilde{u}$ and $\tilde{v}$ are bounded functions on $(0,1)$, A_0 is a constant, $A_1(x,y)$ is a polynomial in x and y of degree 1 and $|q_n^{(1)}(x,y)| \le Cn^{-2}$ for $0 < x, y < 1$. The expression in (4.8) equals x^{-1} times the expresion in (5.4) of [3] corresponding to the case $s = 0$. We proved in [3] (cf. the proof of [3, Proposition 5.1]) that this expression equals

$$u(x,y)P_r(\pi(x-y)) + v(x,y)Q_r(\pi(x-y)),$$

where u and v are bounded functions on $(0,1)\times(0,1)$, plus some terms whose absolute values are bounded by either $C\log\big((2x)/|x-y|\big)$ or $C(2-x-y)^{-1}$.

Each of the aforementioned bounds is controlled by $C|x-y|^{-1}$; in addition, also $P_r(\pi(x-y))$ as well as $|Q_r(\pi(x-y))|$ are bounded by $C|x-y|^{-1}$ for $0<x,y<1$. Taking into account the factor x^{-1} the estimate $|x^{-1}O_{1,0}(x,y)|\le C|x-y|^{-2}$ follows. The bound of $y^{-1}O_{0,1}(x,y)$ is treated in a similar fashion ($E_{0,0,1}$ is involved).

It remains to consider $O_{0,0}(x,y)$. Using once more [3, Lemma 4.1] and [3, Lemma 4.2] with $j=0$, $\gamma=1$ and $\ell=2$ shows that $O_{0,0}(x,y)$ is the sum of four terms of the form

$$u(x)v(y)\sum_{n=N}^{\infty} r^n E_{0,1,2}(n,x,y)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n y),$$

where u and v are bounded functions on $(0,1)$. After applying trigonometric identities and expanding $E_{0,1,2}(n,x,y)$, we treat $O_{0,0}(x,y)$ as the sum of four terms of the form

$$\tilde{u}(x)\tilde{v}(y)\sum_{n=N}^{\infty} r^n n\Big(A_0+\frac{A_1(x,y)}{n}+\frac{A_2(x,y)}{n^2}+q_n^{(2)}(x,y)\Big)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y)), \tag{4.9}$$

where $\tilde{u}$ and $\tilde{v}$ are bounded functions on $(0,1)$, A_0 is a constant, $A_i(x,y)$ are polynomials in x and y of degree i and $|q_n^{(2)}(x,y)|\le Cn^{-3}$ for $0<x,y<1$. The expressions resulting from each summand in the expansion of $E_{0,1,2}(n,x,y)$ other than the first one have already been discussed in the treatment of $O_{1,0}(x,y)$; they are bounded by $C|x-y|^{-1}$. Therefore, we are left with the series

$$\sum_{n=N}^{\infty} r^n n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y)).$$

It is possible to extend the summation in the above series from $n=1$ since

$$\Big|\sum_{n=1}^{N-1} r^n n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x\pm y))\Big|\le\sum_{n=1}^{N-1} n\le CN^2\le Cx^{-2}.$$

The series with the plus sign and summation starting from $n=1$ are easily treated. Indeed, assuming $0<x,y<3/4$, we apply [3, Lemma 3.3] to obtain

$$\Big|\sum_{n=1}^{\infty} r^n n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x+y))\Big|\le C(x+y)^{-2}\le Cx^{-2}.$$

Assuming $3/8<x,y<1$, write $x=1-u$, $y=1-v$. Then $0<u+v<3/2$; hence

$$\begin{aligned}\Big|\sum_{n=1}^{\infty} r^n n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(x+y))\Big| &= \Big|\sum_{n=1}^{\infty} r^n n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(\pi n(u+v))\Big|\\ &\le C(u+v)^{-2}\\ &= C(2-x-y)^{-2},\end{aligned}$$

and the last expression is bounded by $C|x-y|^{-2}$ on the considered region. The series with the minus sign and summation starting from $n = 1$ gives either $\frac{dQ_r}{dt}(\pi(x-y))$ or $\frac{dP_r}{dt}(\pi(x-y))$. Both terms are bounded by $C|x-y|^{-2}$; cf. [3, Lemma 3.1] taken with $n = 1$. The proof of (4.3) is finished □

Proposition 4.2. *Let* $\mu, \nu > -1$. *Then for every* $x \neq y$, $0 < x, y < 1$, *the limit*

$$L(x,y) = \lim_{r\to 1^-} L(r,x,y)$$

exists and satisfies

$$|L(x,y)| \leq C \begin{cases} x^{-\mu-3/2}y^{\mu+1/2}, & 0 < y \leq x/2, \\ |x-y|^{-1}, & x/2 < y < \min\{1, 3x/2\}, \\ x^{\nu+1/2}y^{-\nu-3/2}, & \min\{1, 3x/2\} \leq y < 1, \end{cases}$$

and

$$|\nabla L(x,y)| \leq C|x-y|^{-2}, \quad x/2 < y < \min\{1, 3x/2\}. \tag{4.10}$$

In both cases, C *is independent of* $0 < x < 1$ *and* $0 < y < 1$.

Proof. Once we prove the existence of the limit, the required estimates follow directly from Proposition 4.1. To be precise, justifying (4.10) requires also the identity

$$\frac{\partial}{\partial y}\Big(\lim_{r\to 1^-} L(r,x,y)\Big) = \lim_{r\to 1^-} \frac{\partial}{\partial y} L(r,x,y) \tag{4.11}$$

and similarly for $\frac{\partial}{\partial x}$. Assume for the moment that $\lim_{r\to 1^-} L(r,x,y)$ exists; what is still needed for proving (4.11) is the fact that for fixed x, $0 < x < 1$, the convergence on the right of (4.11) is locally uniform in y. Since

$$\frac{\partial}{\partial y} L(r,x,y) = \frac{2\mu+1}{y} L(r,x,y) - \widetilde{L}(r,x,y)$$

(cf. (4.4) and the notation that follows), it is sufficient to check that for given x, $0 < x < 1$, the convergence of $L(r,x,y)$ and $\widetilde{L}(r,x,y)$ with $r \to 1^-$ is locally uniform in y. For $L(r,x,y)$ this can be explained along the lines of the proof of the existence of $\lim_{r\to 1^-} L(r,x,y)$; see the lines that follow. For $\widetilde{L}(r,x,y)$, the argument is essentially the same; hence we do not provide the details (see the proof of Proposition 4.1).

Using (4.6) with $M = 1$, we expand $(\lambda_{n,\nu}x)^{\frac{1}{2}} J_\nu(\lambda_{n,\nu}x)$ and $(\lambda_{n,\mu}y)^{\frac{1}{2}} J_{\mu+1}(\lambda_{n,\mu}y)$ to get

$$L(r,x,y) = \sum_{j,l=0}^{1} x^{-j}y^{-l} O_{j,l}(r,x,y) + J_1(r,x,y) + J_2(r,x,y) + G(r,x,y).$$

Here the $O_{j,l}$ terms capture the part that comes from the main parts of the aforementioned expansions and are linear combinations of expressions of the form

$$\sum_{n=1}^{\infty} r^n d_{n,\nu} d_{n,\mu} \lambda_{n,\nu}^{-j} \lambda_{n,\mu}^{-l} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\lambda_{n,\nu} x) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\lambda_{n,\mu} y);$$

J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one and hence is a linear combination of terms of the form

$$y^{-\delta} \sum_{n=1}^{\infty} r^n d_{n,\nu} d_{n,\mu} \lambda_{n,\mu}^{-\delta} H_1(\lambda_{n,\nu} x) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\lambda_{n,\mu} y), \quad \delta = 0, 1;$$

J_2 acts as J_1 but with the position of the both expansions switched and hence is a linear combination of terms of the form

$$x^{-\delta} \sum_{n=1}^{\infty} r^n d_{n,\nu} d_{n,\mu} \lambda_{n,\nu}^{-\delta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\lambda_{n,\nu} x) H_1(\lambda_{n,\mu} y), \quad \delta = 0, 1;$$

finally, G captures the part that comes from the remainders,

$$G(r, x, y) = \sum_{n=1}^{\infty} r^n d_{n,\nu} d_{n,\mu} H_1(\lambda_{n,\nu} x) H_1(\lambda_{n,\mu} y).$$

From the bound $H_1(z) = O(z^{-2})$, $z \geq 1$, it is evident that each of the series as in $G(r, x, y)$ or in the terms entering into either J_1 or J_2, but with the factor r^n removed, is absolutely convergent since, for sufficiently large n, either $|H_1(\lambda_{n,\nu} x)| \leq Cn^{-2}$ or $|H_1(\lambda_{n,\mu} y)| \leq Cn^{-2}$ (or both). Thus, the corresponding expressions converge with $r \to 1^-$. In addition, the convergence is locally uniform in y. It is therefore sufficient to analyze the $O_{j,l}$ terms. Given $j, l \in \{0, 1\}$, we use [3, Lemma 4.2] with the given j, $\gamma = -l$ and $\ell = 1$. Then $O_{j,l}$ can be written as a linear combination of terms of the form

$$u(x, y) \sum_{n=1}^{\infty} r^n \Big(A_0 + \frac{A_1(x, y)}{n} + q_n^{(1)}(x, y) \Big) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\pi n (x \pm y)),$$

where $u(x, y)$ is a bounded function, A_0 is a constant, $A_1(x, y)$ is a polynomial and $|q_n^{(1)}(x, y)| \leq Cn^{-2}$. We split the last series onto the three expressions corresponding to A_0, A_1/n and $q_n^{(1)}$. The expression corresponding to $q_n^{(1)}$ converges with $r \to 1^-$ since the series as in this expression with the factor r^n removed is absolutely convergent; in addition, the convergence is locally uniform in y. The first two expressions also converge with $r \to 1^-$, locally uniformly in y. Indeed, the expression that corresponds to A_0 contains either $P_r(\pi(x \pm y))$ or $Q_r(\pi(x \pm y))$, so convergence is obvious; the second expression corresponding to A_1/n also has a compact form (see [12, p. 2]), and the required convergence follows. The proof of the proposition is completed. □

Remark 4.3. In the case $\mu, \nu \geq 1/2$, the kernel $L(x,y)$ satisfies the standard CZ estimates in the whole square $(0,1)\times(0,1)$. In addition, $L(x,y)$ satisfies (3.1) under the weaker assumptions $\mu,\nu \geq -1/2$.

Clearly, the outermost estimates of $|L(x,y)|$ in Proposition 4.2 are stronger than $O(|x-y|^{-1})$ if we assume $\mu,\nu > -1/2$. On the other hand, it may be seen in the proof of Proposition 4.1 for the kernel $L(r,x,y)$ (the estimate for $|L(r,x,y)|$ follows from (4.2), and the corresponding estimate for $|\nabla L(r,x,y)|$ is deduced from (4.4)). However, if either $\mu < 1/2$ or $\nu < 1/2$, the factor $L(r,x,y)y^{-1}$ appearing in (4.4) cannot be controlled by $(x-y)^{-2}$ in the region $0 < y \leq x/2$.

Remark 4.4. For $\mu = -1/2$ and $\nu = 1/2$,

$$L(x,y) = \cos\big(\frac{\pi}{2}x\big)\frac{\sin(\pi y)}{\cos(\pi y) - \cos(\pi x)}.$$

Indeed, a direct calculation shows

$$\begin{aligned} L(r,x,y) =& \cos\big(\frac{\pi}{2}x\big)\sum_{n=1}^{\infty} r^n\Big(\sin\big(\pi n(x+y)\big) - \sin\big(\pi n(x-y)\big)\Big) \\ &+ \sin\big(\frac{\pi}{2}x\big)\sum_{n=1}^{\infty} r^n\Big(\cos\big(\pi n(x-y)\big) - \cos\big(\pi n(x+y)\big)\Big), \end{aligned}$$

Passing to the limit with $r \to 1^-$ then gives

$$L(x,y) = \cos\big(\frac{\pi}{2}x\big)\left(\frac{1}{\tan\left(\pi\frac{x+y}{2}\right)} - \frac{1}{\tan\left(\pi\frac{x-y}{2}\right)}\right),$$

and an application of trigonometric identities does the job. The fact that $L(x,y)$ is a C^1 function on $(0,1)\times(0,1)\setminus\Delta$ and satisfies the estimates of Proposition 4.2 now follows by inspection.

Finally, we show that the kernel $L(x,y)$ is associated with $T_{\mu\nu}$ in the sense of CZ theory.

Proposition 4.5. *Let $f,g \in C_c^\infty(0,1)$ have disjoint supports. Then*

$$\langle T_{\mu\nu}f, g\rangle = \int_0^1\int_0^1 L(x,y)f(y)\overline{g(x)}\,dy\,dx \tag{4.12}$$

Proof. Let $g = \sum_{n=1}^\infty c_n^\nu(g)\psi_n^\nu$. By Parseval's identity,

$$\langle T_{\mu\nu}f, g\rangle = \sum_{n=1}^{\infty} c_n^\mu(f)\overline{c_n^\nu(g)}. \tag{4.13}$$

We show that the right sides of (4.12) and (4.13) coincide. Denoting by $F(x,y)$ the function from Proposition 4.2 that majorizes $|L(x,y)|$, we see that

$$\int_0^1 \int_0^1 |F(x,y) f(y) \overline{g(x)}|\, dy\, dx < \infty.$$

Therefore, the dominated convergence theorem justifies the second identity in the following chain of identities

$$\begin{aligned}
\langle T_{\mu\nu} f, g\rangle &= \int_0^1 \int_0^1 \lim_{r\to 1^-} L(r,x,y) f(y)\overline{g(x)}\, dy\, dx \\
&= \lim_{r\to 1^-} \int_0^1 \int_0^1 L(r,x,y) f(y)\overline{g(x)}\, dy\, dx \\
&= \lim_{r\to 1^-} \int_0^1 T_{\mu\nu,r} f(x)\overline{g(x)}\, dx \\
&= \lim_{r\to 1^-} \sum_{n=1}^{\infty} r^n c_n^{\mu}(f)\overline{c_n^{\nu}(g)}.
\end{aligned}$$

The third identity is explained in the proof of Theorem 1.1 in [3], and the fourth one is a consequence of [3, (1.10)] and Parseval's identity. Finally, since the series $\sum_{n=1}^{\infty} c_n^{\mu}(f)\overline{c_n^{\nu}(g)}$ converges (see [3, Lemma 2.2]) the last limit equals the right side of (4.13). □

5 Proofs of the main results

Recall that $T^1_{\mu\nu}$ and $T^2_{\mu\nu}$ denote the integral operators

$$T^1_{\mu\nu} f(x) = \int_0^{x/2} L(x,y) f(y)\, dy, \quad T^2_{\mu\nu} f(x) = \int_{\min\{1,3x/2\}}^1 L(x,y) f(y)\, dy.$$

By taking $p = 2$ and $w(x) \equiv 1$ in (2.4) and (2.5) it follows that $T^1_{\mu\nu}$ and $T^2_{\mu\nu}$ are bounded on L^2; see the computations in the proof of Theorem 2.1 below. Thus

$$T^3_{\mu\nu} = T_{\mu\nu} - T^1_{\mu\nu} - T^2_{\mu\nu}$$

is also bounded on L^2. Moreover, by Proposition 4.5, $T^3_{\mu\nu}$ is associated with the kernel $L(x,y)\chi_{\mathcal{D}}(x,y)$ which is, by Proposition 4.2, a local CZ kernel (the gradient estimate (4.10) implies smoothness conditions (3.2) and (3.3)). Thus $T^3_{\mu\nu}$ is a local CZ operator.

Proof of Theorem 2.1. Assume that w satisfies the assumptions of the theorem. Using the weighted Hardy inequality (2.4), we obtain

$$\begin{aligned}\int_0^1 |w(x)T^1_{\mu\nu}f(x)|^p\,dx &= \int_0^1 \Big|w(x)\int_0^{x/2} L(x,y)f(y)\,dy\Big|^p\,dx\\ &\le C\int_0^1 \Big(w(x)x^{-\mu-3/2}\int_0^{x/2} y^{\mu+1/2}|f(y)|\,dy\Big)^p\,dx\\ &\le C\int_0^1 |w(x)f(x)|^p\,dx.\end{aligned}$$

Similarly, using the weighted Hardy inequality (2.5), we get

$$\int_0^1 |w(x)T^2_{\mu\nu}f(x)|^p\,dx \le C\int_0^1 |w(x)f(x)|^p\,dx.$$

Finally, the corresponding L^p inequality for $T^3_{\mu\nu}$ is a consequence of Theorem 3.2. Thus (2.9) follows.

To prove (2.10), we first note that the existence of $c^\mu_n(f)$, $f \in L^p(w)$, follows from the assumption made on w. Indeed, we use Hölder's inequality, the estimate [3, (2.8)] and either the fact that the second term in (2.1) for $r = 1/2$ is finite to check that $\int_0^{1/2} |f(x)\varphi^\mu_n(x)|\,dx < \infty$ or the fact that the second term in (2.2) (with μ replacing ν) is finite to verify that $\int_{1/2}^1 |f(x)\varphi^\mu_n(x)|\,dx < \infty$. Similarly, the fact that w satisfies conditions (2.1) and (2.2) guarantees that $\varphi^\nu_n \in L^p(w)$. Indeed, the estimate [3, (2.8)] and either the fact that the first term in (2.1) for $r = 1/2$ is finite or the fact that the second term in (2.2) is finite show that $\int_0^{1/2} |\varphi^\nu_n(x)w(x)|\,dx < \infty$ and $\int_{1/2}^1 |\varphi^\nu_n(x)w(x)|\,dx < \infty$ correspondingly. In the same way, the fact that w satisfies (2.1) and (2.2) ensures the existence of the coefficients $c^\nu_n(T_{\mu\nu}f)$ for any $f \in L^p(w)$, since then also $T_{\mu\nu}f \in L^p(w)$.

To complete the proof of (2.10), we argue as in the proof of (1.5) in [3, Theorem 1.2]. □

Proof of Theorem 2.2. Argue as in the proof of Theorem 2.1, but using (2.14), (2.15) and (2.7) instead of (2.4), (2.5) and (2.6). □

Proof of Theorem 2.3. We use the following multiplier result for the sine expansions, which is a special case of [9, Theorem (9.12)], a corresponding multiplier result for Fourier series.

Proposition 5.1. *Let $1 < p < \infty$, $m \in M(1,1)$ and $w^p \in A_p(0,1)$. Then for any sequence $d = \{d_n\}_{n\ge1}$ with $d_n = 0$ for all but a finitely many n,*

$$\Big\|\sum_{n=1}^\infty m_n d_n \sin(\pi n x)\Big\|_{L^p(w)} \le CB(m,1,1)\Big\|\sum_{n=1}^\infty d_n \sin(\pi n x)\Big\|_{L^p(w)},$$

with C independent of m and d.

This theorem is obtained from Theorem 9.12 of [9] by taking there $s = \lambda = 1$, $l = 0$, $g(x) \equiv 1$, $V(x) = w(|x/\pi|)^p$, $-\pi < x < \pi$ (so that $V \in A_p(-1,1)$), extending the sequence $\{m_j\}_{j=1}^{\infty}$ to the sequence $\{m_j^*\}_{j=-\infty}^{\infty}$ by setting $m_{-j}^* = m_j$ for $j < 0$ and $m_0^* = 0$ (so that $m^* \in M^*(1,1)$ in the notation of [9]), and restricting the class S_{-1} (which appears in [9, Theorem (9.17)]) of all trigonometric polynomials to the class of odd trigonometric polynomials, i.e., functions of the form $f(x) = \sum_{|j|\le N} d_j e^{ijx}$, $d_{-j} = d_j$, $N \ge 0$. Since the hypotheses imposed on w allow to transplant between $\nu \ge -1/2$ and $\mu = 1/2$ back and forth (note that the assumption $w^p \in A_p(0,1)$ implies (2.1) and (2.2) specified to $\mu = \nu = 1/2$), the main statement of Theorem 2.3 now follows from the inequalities

$$\begin{aligned}\Big\| \sum_{n=1}^{\infty} m_n d_n \psi_n^{\nu} \Big\|_{L^p(w)} &\le C_{\nu,1/2} \Big\| \sum_{n=1}^{\infty} m_n d_n \psi_n^{1/2} \Big\|_{L^p(w)} \\ &\le C C_{\nu,1/2} B(m,1,1) \Big\| \sum_{n=1}^{\infty} d_n \psi_n^{1/2} \Big\|_{L^p(w)} \\ &\le C C_{\nu,1/2} B(m,1,1) C_{1/2,\nu} \Big\| \sum_{n=1}^{\infty} d_n \psi_n^{\nu} \Big\|_{L^p(w)}.\end{aligned}$$

where $C_{\mu,\nu}$ denotes the constant appearing in (2.9). The statement of Theorem 2.3 concerning the extension on $L^p(w)$ follows from the fact that the subspace spanned by $\{\psi_n^{\nu} : n = 1, 2, \dots\}$ is dense in $L^p(w)$.

Let us explain this last claim. Since by assumption w satisfies (2.2) and (2.1) with μ replaced by ν, the coefficients $c_n^{\nu}(f)$, $f \in L^p(w)$, exist and $\psi_n^{\nu} \in L^p(w)$, $n = 1, 2, \dots$ (see the proof of Theorem 2.1). What is more important, an immediate adaptation of the proof of [3, Lemma 2.1] shows that $c_n^{\nu}(f) = O(n^{\rho})$ for $\rho = \rho(f, \nu, p, w)$ and $\|\psi_n^{\nu}\|_{L^p(w)} = O(n^{\tau})$ for $\tau = \tau(\nu, p, w)$ (again, the assumptions imposed on w are essential). With these two asymptotics, it is easy to see that the argument used in the proof of [3, Lemma 2.3] can be adapted to show the density of the span of $\{\psi_n^{\nu} : n = 1, 2, \dots\}$ in $L^p(w)$. □

Proof of Proposition 2.4. If $1 < p < \infty$, it is sufficient to check that w verifies (2.1) and (2.2) for $\mu = -1/2, \nu = -1/2$, since the cases $\mu \ge -1/2, \nu \ge -1/2$ then follow. If w satisfies (2.8), then $w^p \in A_p(0,1)$; hence

$$\|wMf\|_{L^p} \le C \|wf\|_{L^p}. \tag{5.1}$$

Since the Hardy operator P_1 is dominated by M,

$$\left| \frac{1}{x} \int_0^x f(t)\, dt \right| \le 2Mf(x), \quad x \in (0,1),$$

it follows that

$$\left\| \frac{w(x)}{x} \int_0^x f(t)\,dt \right\|_{L^p} \le C\|w(x)Mf(x)\|_{L^p}. \tag{5.2}$$

This is (2.4) for $\mu = -1/2$; hence w satisfies (2.1) with $\mu = -1/2$. On the other hand, if w satisfies (2.8), then $w^{-p'} \in A_{p'}(0,1)$; hence (5.1) is satisfied with w^{-1} and p' replacing w and p. Thus (5.2) holds with the analogous replacement. It is easily seen that the dual inequality to (5.2) with the aforementioned replacement is (2.5) with $\nu = -1/2$; hence w satisfies (2.2) with $\nu = -1/2$.

In the case $p = 1$, the argument is similar. If w satisfies (2.8) with $p = 1$, then $w \in A_1(0,1)$; hence

$$\|Mg\|_{L^{1,\infty}(w)} \le C\|g\|_{L^1(w)}.$$

Consequently, given $\mu \ge -1/2$, it follows that

$$|P_{\mu+3/2}f(x)| \le |P_1(f(t)t^{-(\mu+1/2)})(x)| \le 2M\big(f(t)t^{-(\mu+1/2)}\big)(x);$$

therefore,

$$\big\|P_{\mu+3/2}f\big\|_{L^{1,\infty}(w)} \le C\big\|f(x)x^{-(\mu+1/2)}\big\|_{L^1(w)}.$$

This is (2.14); hence w satisfies (2.11). On the other hand, if w satisfies (2.8) with $p = 1$, then

$$\frac{1}{v-u}\int_u^v w \le C \operatorname*{ess\,inf}_{x\in(u,v)} w(x), \qquad 0 \le u < v \le 1.$$

This is the A_1 condition, readily seen to be equivalent to $Mw(x) \le Cw(x)$ a.e. (cf. [4, p. 134]), which is necessary and sufficient for

$$\|Mgw^{-1}\|_{L^\infty} \le C\|gw^{-1}\|_{L^\infty}$$

to hold (cf. [6, Theorem 4]). Hence, given $\nu \ge -1/2$, we obtain

$$\big\|P_{\nu+3/2}fw^{-1}\big\|_{L^\infty} \le C\big\|f(x)x^{-(\nu+1/2)}w(x)^{-1}\big\|_{L^\infty}. \tag{5.3}$$

Since the dual to $L^1(w)$ is $L^\infty(w^{-1})$ (with the pairing $h \mapsto \int_0^1 h\varphi$, $\varphi \in L^\infty(w^{-1})$!), it is easily seen that the dual inequality to (5.3) is

$$\big\|Q_{-(\nu+1/2)}f\big\|_{L^1(w)} \le C\big\|f(x)x^{\nu+3/2}\big\|_{L^1(w)},$$

which implies (2.15); thus (2.13) when $\nu > -1/2$ or (2.12) when $\nu = -1/2$ follow. (Note that for $\nu = -1/2$, (2.12) implies (2.13) with any $\delta > 0$; however, (2.13) was claimed only for $\nu > -1/2$.) □

Proof of Proposition 2.5. One of the two required inclusions is contained in Proposition 2.4. Checking the other inclusion, we note that it can be easily read off from the proof of Theorem 2.1. In fact, a slight generalization of it holds in the case when w satisfies (2.1) with $\mu = -1/2$, (2.2) with $\nu = -1/2$, and (2.3). Specifically, the theorem is valid for T_K replacing $T_{\mu\nu}$, provided that $K(x, y)$ is a kernel on $(0,1) \times (0,1) \setminus \Delta$ satisfying the growth condition (3.1) globally (i.e., for all $(x, y) \in (0,1) \times (0,1) \setminus \Delta$), the gradient condition (3.4) locally (i.e. in the region $\{(x, y) \in (0,1) \times (0,1) \setminus \Delta : x/2 < y < 3x/2\}$), and the associated operator T_K is bounded on L^2.

Thus, let $\mu = \nu = -1/2$ and let w be a weight satisfying (2.1) with $\mu = -1/2$, (2.2) with $\nu = -1/2$, and (2.3). The kernels

$$K_o(x, y) = \frac{2}{\pi} \frac{y}{y^2 - x^2}, \quad K_e(x, y) = \frac{2}{\pi} \frac{x}{y^2 - x^2},$$

considered on $(0,1) \times (0,1) \setminus \Delta$ are easily seen to satisfy the growth and gradient conditions (3.1) and (3.4) (even globally!); hence the generalization mentioned above applies to the associated operators. These operators are restrictions of the Hilbert transform to odd and even functions considered only on the interval $(-1, 1)$ (hence their boundedness on L^2 follows); see [1] for the Hilbert transform case, as well as the case of the conjugate operator. It therefore follows from [1, Theorems 1 and 2] (or rather their straightforward generalizations) that $W(x) = w(x)^p$ satisfies the inequalities in points (a) of these theorems (so called A_p^o and A_p^e conditions), of course restricted to $(0, 1)$. A remark made in the middle of [1, p. 101] then shows that $W(x) = w(x)^p$ satisfies the A_p condition on $(0, 1)$, i.e., the condition (2.8). □

REFERENCES

[1] K. F. Andersen, *Weighted norm inequalities for Hilbert transforms and conjugate functions of even and odd functions*, Proc. Amer. Math. Soc. **56** (1976), 99–107.

[2] K. F. Andersen and B. Muckenhoupt, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, Studia Math. **72** (1982), 9–26.

[3] Ó. Ciaurri and K. Stempak, *Transplantation and multiplier theorems for Fourier-Bessel expansions*, Trans. Amer. Math. Soc. **358** (2006), 4441-4465.

[4] J. Duoandikoetxea, *Fourier Analysis*, Amer. Math. Soc., Providence, RI, 2001.

[5] N. N. Lebedev, *Special Functions and Thier Applications*, Dover, New York, 1972.

[6] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[7] B. Muckenhoupt, *Hardy's inequalities with weights*, Studia Math. **44** (1972), 31–38.

[8] B. Muckenhoupt, *Transplantation theorems and multiplier theorems for Jacobi series*, Mem. Amer. Math. Soc. **356**, 1986.

[9] B. Muckenhoupt, R. L. Wheeden and W.-S. Young, *Sufficiency conditions for L^p multipliers with general weights*, Trans. Amer. Math. Soc. **300** (1987), 463–502.

[10] A. Nowak and K. Stempak, *Weighted estimates for the Hankel transform transplantation operator*, Tohoku Math. J. **58** (2006), 277–301.

[11] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1958.

[12] A. Zygmund, *Trigonometric Series*, Vol. I, 2nd ed., Cambridge University Press, Cambridge, 1959.

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(Received October 9, 2005 and in revised form April 25, 2006)