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On Some Algebras Related to Simple Lie Triple Systems

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The set $\mathrm{Hom}_{\mathrm{Der}\,T}(T \otimes_F T, T)$ is determined for any simple finite dimensional Lie triple system T over a field F of characteristic zero. It turns out that it contains nontrivial elements if and only if T is related to a simple Jordan algebra. In particular this provides a new proof of the determination by Laquer of the invariant affine connections in the simply connected compact irreducible Riemannian symmetric spaces.

1. INTRODUCTION

Given a Lie group G acting smoothly and transitively on a manifold M, the isotropy subgroup H at any point $p \in M$ (so that $M \simeq G/H$), the Lie algebra $\mathfrak{g}$ of the group G and the Lie algebra $\mathfrak{h}$ of H ($\mathfrak{h} \leq \mathfrak{g}$), the manifold M is called a *reductive homogeneous space* in case

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{$\sharp$}$$

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for a subspace $\mathfrak{m}$ with $(\operatorname{Ad} h)(\mathfrak{m}) \subset \mathfrak{m}$ for any $h \in H$, which implies that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ (and if H is connected, these are equivalent).

In this situation Nomizu [19, Theorem 8.1] established a bijection between the set of G-invariant affine connections on M and the set of bilinear maps

$$\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$$

with the property that

$$\alpha\big((\operatorname{Ad} h)X, (\operatorname{Ad} h)Y\big) = (\operatorname{Ad} h)\,\alpha(X, Y) \qquad \forall h \in H,\ \forall X, Y \in \mathfrak{m},$$

that is, with $\operatorname{Ad} H|_{\mathfrak{m}} \subseteq \operatorname{Aut}(\mathfrak{m}, \alpha)$ (the automorphism group of the nonassociative algebra $(\mathfrak{m}, \alpha)$). Again the condition $\operatorname{Ad} H|_{\mathfrak{m}} \subseteq \operatorname{Aut}(\mathfrak{m}, \alpha)$ implies the condition $\operatorname{ad} \mathfrak{h}|_{\mathfrak{m}} \subseteq \operatorname{Der}(\mathfrak{m}, \alpha)$, and they are equivalent if H is connected. The torsion, curvature, geodesics, holonomy, etc. of the connection associated to $(\mathfrak{m}, \alpha)$ can then be expressed and studied in terms of this nonassociative algebra.

Notice too that the set of the multiplications $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that $\operatorname{ad} \mathfrak{h}|_{\mathfrak{m}} \subseteq \operatorname{Der}(\mathfrak{m}, \alpha)$ is nothing else but the vector space of the homomorphisms of $\mathfrak{h}$-modules from $\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}$ into $\mathfrak{m}$: $\operatorname{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$, which is a purely algebraic object.

Thus, the problem of determining the invariant affine connections on reductive homogeneous spaces reduces to the problem of classifying algebras with a given subgroup of its automorphism group or a given subalgebra of its Lie algebra of derivations. The same problem has arisen in studying real division algebras [1], nonunital composition algebras [2, 20] or in the classification of some flexible Lie-admissible algebras (see [17] and the references therein), a subject we will come back to later on.

In some cases, the algebras that appear related to some reductive homogeneous spaces are already known algebras. This is what happens for the six- and seven-dimensional spheres, viewed as $S^6 = G_2/SU(3)$ and $S^7 = \operatorname{Spin}(7)/G_2$, where algebras related to a so-called color algebra [3] and the simple non-Lie Malcev algebras [4] appear.

In 1992, Laquer [13, 14] computed the sets $\operatorname{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$ associated with the simply connected compact irreducible Riemannian symmetric spaces (see [7] for definitions). For these spaces, the decomposition (♯) is a grading of $\mathfrak{g}$ over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, $\mathfrak{m}$ thus being a (simple) Lie triple system. Therefore the problem was to compute $\operatorname{Hom}_S(T \otimes_{\mathbb{R}} T, T)$ for some $\mathbb{Z}_2$-graded simple (that is, without nontrivial homogeneous ideals) finite dimensional real Lie algebra $L = S \oplus T$ (S being the even part and T the odd one). This was done by extending scalars to the complex field and by decomposing there the tensor product $T_{\mathbb{C}} \otimes_{\mathbb{C}} T_{\mathbb{C}}$ as a direct sum of irreducible modules. The vector space $\operatorname{Hom}_S(T \otimes_{\mathbb{R}} T, T)$ turned out to be

0 (which amounts to saying that the only invariant affine connection is the canonical one) with a few exceptions.

Our purpose in this paper is twofold:

(i) We will determine the set

$$\mathrm{Hom}_S(T \otimes_F T, T)$$

for any $\mathbb{Z}_2$-graded simple finite dimensional Lie algebra $L = S \oplus T$ over any arbitrary field of characteristic zero, without actually decomposing $T \otimes_F T$ (when restricted to the real field, this determines the invariant affine connections on the irreducible affine symmetric spaces).

(ii) We will relate the resulting nontrivial algebras with the simple finite dimensional Jordan algebras.

In a sense, we can say that the Jordan algebras are responsible for the existence of noncanonical invariant affine connections on the symmetric spaces. The close relationship of Jordan algebras with some symmetric spaces is well known (see, for instance, [16, Chapter VIII]). Our results give another aspect of this relationship.

Actually, our determination of the sets $\mathrm{Hom}_S(T \otimes_F T, T)$ over algebraically closed fields will be very quick, using information on the $\mathbb{Z}_2$-graded simple Lie algebras that is contained in the Dynkin diagrams associated with the corresponding Lie triple systems T by Faulkner [5], with no need of a case-by-case decomposition of $T \otimes_F T$. Our approach is inspired by [6, Theorem 1]. This allows us to simplify some of the results in [13] and [14].

From the point of view of nonassociative algebras, a central role in what follows is played by the multiplication defined on the set J_0 of (generic) trace zero elements in a Jordan algebra J, given by projecting the product in J onto J_0 (see $(*)$ in 3.2). This procedure is not new; for example, when applied to Cayley–Dickson algebras it gives all the central simple non-Lie Malcev algebras; it appears too in the construction of the pseudo-octonion algebras (see [17] and the references therein).

2. LIE TRIPLE SYSTEMS

In this section, some facts concerning simple Lie triple systems and the Dynkin diagrams attached to them will be recalled.

2.1

A Lie triple system is a vector space T, over a ground field F, equipped with a trilinear product $[xyz]$ satisfying

$$[xxy] = 0$$
$$[xyz] + [yzx] + [zxy] = 0$$
$$\text{the map } [xy-]\colon T \to T,\ z \mapsto [xyz] \text{ is a derivation of } T$$

for any $x, y, z \in T$. If $S = \operatorname{span}\langle [xy-\ ] : x, y \in T\rangle$, S is a Lie subalgebra of the Lie algebra $gl(T)$ (actually of $\operatorname{Der}(T)$); the $\mathbb{Z}_2$-graded Lie algebra $L = S \oplus T$ (where the multiplication is given by the multiplication in S (even part) as a subalgebra of $gl(T)$, the natural action of S on T and $[t_1, t_2] = [t_1 t_2 -\ ]$ for any $t_1, t_2 \in T$) is called the *standard imbedding* of T. Thus any Lie triple system is nothing else but the odd part of a $\mathbb{Z}_2$-graded Lie algebra with product $[xyz] = [[x, y], z]$. The Lie triple system is simple if and only if its standard imbedding is graded simple (for these facts see [15]).

2.2

Now assume that T is a simple Lie triple system over the field F and let $\Gamma = \{\alpha \in \operatorname{End}_F(T) : \alpha[xyz] = [(\alpha x)yz] = [x(\alpha y)z] = [xy(\alpha z)]\ \forall x, y, z \in T\}$ be its centroid. Γ is a field extension of F and, for any $x, y \in T$,$[xy-\ ] \in \operatorname{End}_\Gamma(T)$; therefore the standard imbedding L is a Γ-algebra. Moreover, the centroid Λ of L is a commutative $\mathbb{Z}_2$-graded algebra: $\Lambda = \Lambda_{\bar 0} \oplus \Lambda_{\bar 1}$, with $\Lambda_{\bar 0} = \{\alpha \in \Lambda : \alpha S \subseteq S,\ \alpha T \subseteq T\}$ and $\Lambda_{\bar 1} = \{\alpha \in \Lambda : \alpha S \subseteq T,\ \alpha T \subseteq S\}$, with any nonzero homogeneous element being invertible (by the graded simplicity of L), and it easily follows that $\Lambda_{\bar 0} = \Gamma$. There are two possibilities:

2.2(a) If there is an element $0 \neq u \in \Lambda_{\bar 1}$, then $\Lambda_{\bar 1} = \Gamma u$, so that $\Lambda = \Gamma[u]$, with $0 \neq u^2 = \lambda \in \Gamma$, $T = uS$, and $L = \Gamma[u] \otimes_\Gamma S$. In this case S is a central simple Lie algebra over Γ and $T = uS$ is the adjoint module for S; so we will refer to this case as T being of *adjoint type*. For any $x, y, z \in S$, $[(ux)(uy)(uz)] = u^2(u[[x, y], z]) = \lambda(u[[x, y], z])$; so T is isomorphic to the Lie triple system S under the product $\{xyz\} = \lambda[[x, y], z]$. Notice that if $u^2 \in \Gamma^2$, then λ can be taken to be 1.

2.2(b) $\Lambda = \Gamma$. Then L is a simple algebra (ungraded): otherwise for any proper ideal I of L, by graded simplicity $(S \cap I) \oplus (T \cap I) = 0$ and $\pi_S(I) \oplus \pi_T(I) = L$ (where π_S and π_T denote the projections on S and T respectively); therefore $S \cap I = T \cap I = 0$ and $\pi_S(I) = S$, $\pi_T(I) = T$. Hence for any $s \in S$ there is a unique $t_s \in T$ with $s + t_s \in I$, and also for any $t \in T$ there is a unique, $s_t \in S$ with $s_t + t \in I$. Then the odd map given by $s \mapsto t_s$ and $t \mapsto s_t$ is easily checked to be an odd element in the centroid of L, a contradiction with $\Lambda = \Gamma$.

Therefore in this case, for any extension field K/Γ, $K \otimes_\Gamma L$ is a simple algebra (and $\mathbb{Z}_2$-graded).

From now on all the algebras and systems considered will be assumed to be finite dimensional.

2.3

Lister showed in [15, Theorem 4.5] (see also [5, Proposition 1]) that, given any simple Lie triple system T over an algebraically closed field of characteristic zero with standard imbedding $L = S \oplus T$, either

2.3(i) S is semisimple and T is an irreducible self-dual S-module (this is always the situation for the adjoint type cases) or

2.3(ii) $S = Fz \oplus [S, S]$, where z is a nonzero central element of S and $[S, S]$ is semisimple, and $T = T_1 \oplus T_2$, where T_1 and T_2 are irreducible dual S-modules, so that z acts as a nonzero scalar on T_1 and as its negative on T_2. This is the situation that occurs after complexifying the simple Lie triple systems associated to the hermitian symmetric spaces.

2.4

Faulkner [5] associated a diagram with any such simple Lie triple system, starting with the Dynkin diagram of the semisimple Lie algebra $[S, S]$ and adding "marked" nodes representing the lowest weight of the representation of $[S, S]$ on each irreducible summand of T. These diagrams thus contain the information on the structure of S and of T as an S-module (which is enough to determine L). The resulting diagrams are exactly the affine diagrams $X_N^{(1)}$ and $X_N^{(2)}$ (which are equipped with some numerical labels in the nodes representing the linear dependence of the roots and weights involved; see [12, Chap. 4]) with one or two marked nodes. More precisely, the diagrams that appear are exactly:

2.4(I) The diagrams $X_N^{(1)}$ with a node labeled by 1 marked (by the symmetry of these diagrams, in this case it does not matter which is the marked node)—these correspond to the adjoint types.

2.4(II) The diagrams $X_N^{(1)}$ with two nodes labeled by 1 marked—these correspond to case 2.3(ii) above. Here the same diagram may give different possibilities according to which two nodes are marked. The same comment applies to the next cases.

2.4(III) The diagrams $X_N^{(1)}$ with a node labeled by 2 marked—in this case the highest weight λ of T as an S-module (which is the negative of the lowest weight since T is self-dual) is not in the root lattice, which is equivalent to 0 not being a weight of the representation of S on T (see [8, Exercise 21.3]).

2.4(IV) The diagrams $X_N^{(2)}$ (so X_N is A_N $(N \geq 2)$, D_N, or E_6) with a node labeled by 1 marked—in this case 0 is a weight of the representation of S on T. This implies that there are roots which are weights and, therefore, at least the short roots of every simple ideal of S are weights (notice that the action of S on T is faithful, so that, for any simple ideal I of S and any $0 \neq v \in T$ in the zero weight space, $Iv \neq 0$).

In case 2.4(I) $L = S \oplus T$ is isomorphic (as ungraded algebra) to the direct sum of two copies of the simple algebra of type X_N. In the remaining cases L is a simple (ungraded) Lie algebra of type X_N and the semisimple Lie algebra $[S, S]$ (which is the whole S if only one marked node appears) is the semisimple Lie algebra associated with the finite Dynkin diagram obtained by removing the marked nodes.

Remark. Faulkner obtained these diagrams just for the Dynkin diagram of $[S, S]$ plus the lowest weights of the irreducible subrepresentations of $[S, S]$ on T. V. Kac had previously arrived at the same situation in his research on automorphisms of finite order in semisimple Lie algebras over an algebraically closed field of characteristic zero (see [10; 11; 12, Chapter 8; 7, Chap. X]), since a $\mathbb{Z}_2$-graded Lie algebra is just a Lie algebra with an automorphism of order 2 (characteristic not 2).

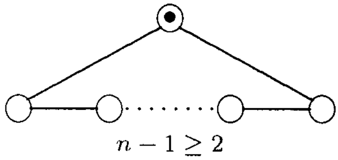
2.5

Among the simple Lie triple systems, the ones obtained from Jordan algebras will be of particular interest to us. Given any simple Jordan algebra J of degree $n \geq 3$ with generic trace t [9, Chap. VI] and multiplication $x \circ y$ over a field of characteristic zero, its set of trace zero elements $J_0 = \{x \in J : t(x) = 0\}$ becomes a simple Lie triple system under the trilinear multiplication $[xyz] = (y \circ z) \circ x - y \circ (z \circ x)$ (the associator of y, z, and x).

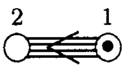
Since $[xy-] = [R_x, R_y]$, where $R_u v = u \circ v = v \circ u$, it follows that in this case $S = [R_J, R_J] = \operatorname{Der} J$ is the Lie algebra of derivations of the Jordan algebra J and the standard imbedding is $L = \operatorname{Der} J \oplus J_0 \cong \operatorname{Der} J \oplus R_{J_0}$, which is the derived subalgebra of the Lie multiplication algebra of J [9, Theorem 8.3].

Over an algebraically closed field F of characteristic zero there are the following possibilities for the simple Jordan algebras of degree $n \geq 3$:

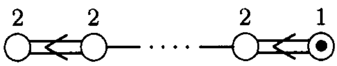
2.5(i) $J = \mathrm{Mat}_n(F)^+$ is the algebra of $n \times n$ matrices over F under the product $x \circ y = \frac{1}{2}(xy + yx)$. In this case $J_0 = sl(n, F)$ is a simple Lie triple system of adjoint type since $\mathrm{Der}\, J = \mathrm{ad}\, J_0$, so that the associated diagram is

$A_{n-1}^{(1)}$ 

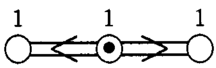
2.5(ii) J is the algebra of symmetric $n \times n$ matrices over F: $J = H(\mathrm{Mat}_n(F), t)$ (t the transposition). In this case $\mathrm{Der}\, J$ is isomorphic to the set of skew-symmetric matrices (see [9, Theorem 6.9]) and the Lie algebra $L = \mathrm{Der}\, J \oplus J_0$ is isomorphic to $sl(n, F)$. The associated diagram is

$A_2^{(2)}$ 

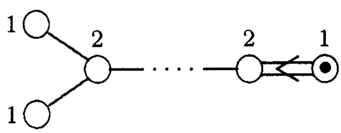
if $n = 3$,

$A_{2l}^{(2)}$ 

if $n = 2l + 1$, $l \geq 2$,

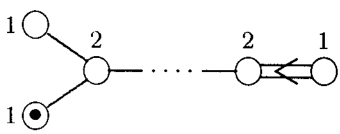
$D_3^{(2)}$ 

if $n = 4$,

$A_{2l-1}^{(2)}$ 

if $n = 2l$, $l \geq 3$.

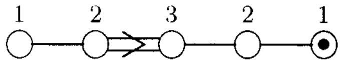
2.5(iii) J is the algebra of symmetric $2n \times 2n$ matrices with respect to the standard symplectic involution. Again in this case $\mathrm{Der}\, J$ is isomorphic to the skew-symmetric matrices with respect to this involution [9, Theorem 6.9], L is isomorphic to $sl(2l, F)$, and the associated diagram is

$A_{2l-1}^{(2)}$ 

$(l \geq 3)$

2.5(iv) J is the exceptional simple Jordan algebra, whose derivation algebra is the Lie algebra of type F_4 [9, Sect. 9.11]. The associated diagram is then

$E_6^{(2)}$



3. ALGEBRAICALLY CLOSED FIELDS

We intend to determine in this section the vector space $\mathrm{Hom}_S(T \otimes_F T, T)$ for any simple Lie triple system T with standard imbedding $L = S \oplus T$, over an algebraically closed field F of characteristic zero, an assumption we will keep throughout the section.

3.1

In case $T = T_1 \oplus T_2$ is a sum of two irreducible S-modules (diagrams 2.4(II)), the center of S is one-dimensional: $Z(S) = Fz$, with $\mathrm{ad}\, z|_{T_1} = 1$, $\mathrm{ad}\, z|_{T_2} = -1$. Therefore $T \otimes_F T$ is the sum of the eigenspaces $2, 0, -2$ for $\mathrm{ad}\, z$, which forces $\mathrm{Hom}_S(T \otimes_F T, T) = 0$.

3.2

In case T is of adjoint type, then $\mathrm{Hom}_S(T \otimes_F T, T) \cong \mathrm{Hom}_S(S \otimes_F S, S)$ and there is a distinguished (skew-symmetric) element in this latter space: the Lie multiplication in S. So at least $\dim \mathrm{Hom}_S(T \otimes_F T, T) \geq 1$. Actually, by [6, Theorem 1 and Corollary 2]

$$\dim \mathrm{Hom}_S(T \otimes_F T, T) = \begin{cases} 2 & \text{if } S \text{ is of type } A_n \ (n \geq 2) \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, if $T = J_0$ is the triple system associated with a simple Jordan algebra of degree $n \geq 3$ with multiplication $x \circ y$, then $S = \mathrm{Der}\, J$ and we obtain a nonzero (symmetric) element of $\mathrm{Hom}_S(T \otimes_F T, T)$ by means of

$$\begin{aligned} J_0 \otimes_F J_0 &\to J_0 \\ x \otimes y &\mapsto x \cdot y = x \circ y - \frac{1}{n} t(x \circ y) 1 \end{aligned} \qquad (*)$$

(the projection of the element $x \circ y$ onto J_0). Notice that if the degree were 2, this would be zero. Besides, if $J = \mathrm{Mat}_n(F)^+$, then the triple system J_0 is both of adjoint type (A_{n-1}) and of "Jordan type."

Therefore, for the adjoint type (diagrams 2.4(I)), $\mathrm{Hom}_S(T \otimes_F T, T)$ is spanned by the "Lie bracket in S" for types other than A_n $(n \geq 2)$ and by the "Lie bracket in S" and the "Jordan product $(*)$" for types A_n $(n \geq 2)$.

3.3

The situation for the simple Lie triple systems with diagrams 2.4(III) can be settled immediately with the following lemma (inspired by [6, Theorem 1]):

LEMMA. *Let S be a semisimple Lie algebra, H a Cartan subalgebra of S, V an irreducible self-dual module with highest weight λ relative to H, and v_λ and $v_{-\lambda}$ nonzero weight vectors for λ and $-\lambda$. For any root α of S relative to H let S_α be the corresponding root space. If V_0 is the weight 0-space in V, then the linear map*

$$\phi: \mathrm{Hom}_S(V \otimes_F V, V) \to \{v \in V_0 : S_\alpha v = 0 \ \forall \alpha \perp \lambda\}$$

$$\varphi \mapsto \varphi(v_\lambda \otimes v_{-\lambda})$$

is well defined and one-to-one.

Proof. Since V is self-dual, $-\lambda$ is the lowest weight of V and $v_\lambda \otimes v_{-\lambda}$ generates $V \otimes_F V$. Hence any $\varphi \in \mathrm{Hom}_S(V \otimes_F V, V)$ is determined by $\varphi(v_\lambda \otimes v_{-\lambda}) \in V_0$. But, for any root α orthogonal to the weight λ and any $x_\alpha \in S_\alpha$, $x_\alpha v_\lambda = 0 = x_\alpha v_{-\lambda}$, so $x_\alpha(v_\lambda \otimes v_{-\lambda}) = 0$ and $x_\alpha \varphi(v_\lambda \otimes v_{-\lambda}) = 0$. ▮

As a direct consequence, $\mathrm{Hom}_S(T \otimes_F T, T) = 0$ for the simple Lie triple systems with diagrams 2.4(III), for which 0 is not a weight of the representation of S on T.

3.4

From the paragraphs above, *only the simple Lie triple systems with diagrams* 2.4(IV) *remain to be studied*. According to Lemma 3.3, we will try to bound the dimension of $\{t \in T_0 : S_\alpha t = 0 \ \forall \alpha \perp \lambda\}$ for a triple system T which is an irreducible S-module of highest weight λ.

LEMMA. *Let T be a simple Lie triple system which is an irreducible module for $S = [TT-]$. Let H be a Cartan subalgebra of the semisimple Lie algebra S, Φ the associated root system, Δ a base of Φ, and Λ the set of weights of T relative to H. For any $\emptyset \neq \Delta' \subseteq \Delta$, let $S_{\Delta'}$ be the subalgebra of S generated by the root spaces S_α, $\alpha \in \Delta' \cup -\Delta'$, let $T_{\Delta'}$ be the sum of the weight spaces T_μ for $\mu \in \Lambda \cap \mathbb{Z}\Delta'$, and let $L_{\Delta'} = S_{\Delta'} \oplus T_{\Delta'}$.*

Then $L_{\Delta'}$ is a graded subalgebra of $L = S \oplus T$ that decomposes as

$$L_{\Delta'} = Z_{\Delta'} \oplus [L_{\Delta'}, L_{\Delta'}],$$

where $Z_{\Delta'} = \{v \in T_0 : S_\alpha v = 0 \ \forall \alpha \in \Delta' \cup -\Delta'\}$ is its center and $[L_{\Delta'}, L_{\Delta'}]$ is semisimple (hence a direct sum of graded simple algebras).

In particular, the codimension in T_0 of $Z_{\Delta'}$ equals the dimension of the zero weight space in $[S_{\Delta'}, T_{\Delta'}]$ (the odd part of $[L_{\Delta'}, L_{\Delta'}]$).

Proof. Since $S_{\Delta'} = \sum_{\alpha \in \Delta'} F h_\alpha + \sum_{\beta \in \Phi \cap \mathbb{Z}\Delta'} S_\beta$ (the h_α's as in [8, p. 37]), $S_{\Delta'}$ is a semisimple subalgebra of S with Cartan subalgebra $H_{\Delta'} = H \cap S_{\Delta'} = \sum_{\alpha \in \Delta'} F h_\alpha$. Besides, Δ' is a base of the root system of $S_{\Delta'}$: $\Phi \cap \mathbb{Z}\Delta'$. Moreover, $L_{\Delta'}$ is clearly a graded subalgebra of L. By complete reducibility $T_{\Delta'} = Z_{\Delta'} \oplus [S_{\Delta'}, T_{\Delta'}]$, and if $0 \neq W$ is an irreducible $S_{\Delta'}$-module in $[S_{\Delta'}, T_{\Delta'}]$, its highest weight belongs to $\mathbb{Z}\Delta'$, so that 0 is a weight in W [8, Exercise 21.3]. Now, for any $0 \neq w \in T_0 \cap W$, $[Z_{\Delta'}, w] \subseteq [T_0, T_0] = 0$, since $H + T_0$ is a Cartan subalgebra (hence abelian) of the semisimple Lie algebra $L = S \oplus T$ ([15, Lemma 4.12] or [12, Lemma 8.1.b)]). Since W is irreducible, it follows that $[Z_{\Delta'}, W] = 0$, so that $Z_{\Delta'}$ is precisely the center of $L_{\Delta'}$. Besides, any abelian ideal of $L_{\Delta'}$ is easily shown to be contained in $Z_{\Delta'}$, so that $[L_{\Delta'}, L_{\Delta'}]$ is semisimple, as required.

3.5

Let T be a simple Lie triple system, let $L = S \oplus T$ be its standard imbedding, and assume that T is an irreducible S-module. As before, we take H a Cartan subalgebra of S, so that the sum of H and the zero weight space T_0 of T is a Cartan subalgebra of L. Let Φ be the root system of S and Λ the set of weights of T with respect to H. A look at the marked diagrams shows that:

3.5(i) If S is simple of type A_1 and 0 is a weight of T, then $\dim T_0 = 1$.

Actually, either T is of adjoint type or the diagram is the one associated with $A_2^{(2)}$; so $L \cong sl(3)$ and $\dim T_0 = \operatorname{rank} L - \operatorname{rank} S = 2 - 1 = 1$.

3.5(ii) If S is simple of type A_2 and 0 is a weight, then T is of adjoint type and $\dim T_0 = 2$.

3.5(iii) If S is simple of type B_l ($l \geq 2$) and $\Phi \subseteq \Lambda$ then $\dim T_0 = l$.

In fact $\Phi \subseteq \Lambda$ if and only if the difference between the highest weight and the longest root is a sum of positive roots. In this case either T is of adjoint type or its diagram is obtained from $A_{2l}^{(2)}$, hence the result ($2l - l = l$).

3.5(iv) There is no simple Lie triple system where S is a direct sum of two simple ideals and all nonzero weights are roots; because for such S, T is a tensor product of an irreducible module for the first ideal times an irreducible module for the second, and the maximal weight cannot be a root.

3.5(v) If S is simple of type C_l $(l \geq 2)$ and $\Lambda = \{\text{short roots}\} \cup \{0\}$, then $\dim T_0 = l - 1$. This is because the only possibilities are given by the affine diagrams $A_{2l-1}^{(2)}$ $(l \geq 3)$ and $D_3^{(2)}$ (for $l = 2$).

3.5(vi) If S is simple of type D_l $(l \geq 3$ with $D_3 = A_3)$ and 0 is a weight, then $\dim T_0 = l - 1$ (the only possibility is given by the affine diagrams $A_{2l-1}^{(2)}$).

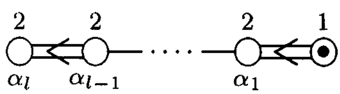
3.6

With this in mind we can tackle the problem of bounding the dimension of

$$T_0^\lambda = \{t \in T_0 : S_\alpha t = 0 \ \forall \alpha \perp \lambda\}$$

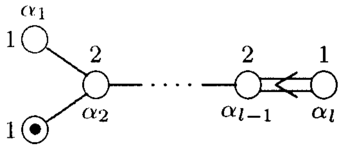
(λ being the highest weight) for the triple systems with diagrams in 2.4(IV). In each case we will find a subset Δ' of the base Δ such that all the roots in Δ' are orthogonal to λ, so that $T_0^\lambda \subseteq Z_{\Delta'}$ and Lemma 3.4 will apply.

- $A_2^{(2)}$. Here $\dim T_0 = 1$ and we already know that $\dim \operatorname{Hom}_S(T \otimes_F T, T) \geq 1$ by 3.2 and 2.5(ii); hence $\dim \operatorname{Hom}_S(T \otimes_F T, T) = \dim T_0 = 1$.
- $A_{2l}^{(2)}$ $(l \geq 2)$



Here all the roots are weights. With $\Delta' = \{\alpha_2, \ldots, \alpha_l\}$, $S_{\Delta'}$ is simple of type B_{l-1} if $l \geq 3$ or A_1 if $l = 2$. From 3.5(iii) $(l \geq 3)$ or 3.5(i) $(l = 2)$ and Lemma 3.4 it follows that $\operatorname{codim} Z_{\Delta'} = l - 1$; so, since $\dim T_0 = 2l - l = l$, it follows that $\dim T_0^\lambda \leq 1$ and we already know that $\dim \operatorname{Hom}_S(T \otimes_F T, T) \geq 1$ in this case (2.5(ii)), so that again $\dim \operatorname{Hom}_S(T \otimes_F T, T) = 1$.

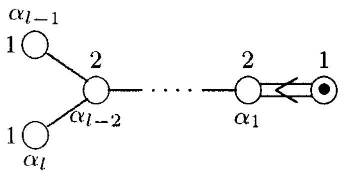
- $A^{(2)}_{2l-1}$ ($l \geq 3$) marked as



Here the nonzero weights are the short roots. We take $\Delta' = \{\alpha_1, \alpha_3, \ldots, \alpha_l\}$ if $l \geq 4$; then because of 3.5(iv) $[L_{\Delta'}, L_{\Delta'}]$ is the sum of two graded simple ideals and by 3.5(i) and 3.5(v) codim $Z_{\Delta'} = 1 + l - 3 = l - 2$; so $\dim T_0^\lambda \leq 1$ ($\dim T_0 = l - 1$). Also, if $l = 3$ we take $\Delta' = \{\alpha_1\}$ (since α_l is not a weight) and then $S_{\Delta'}$ is of type A_1 and by 3.5(i) codim $Z_{\Delta'} = 1$ and $\dim T_0^\lambda \leq 1$.

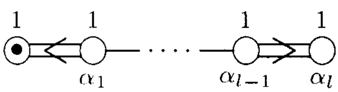
Again by 2.5 we get that $\dim \mathrm{Hom}_S(T \otimes_F T, T) = 1$ in this case.

- $A^{(2)}_{2l-1}$ ($l \geq 3$) marked as



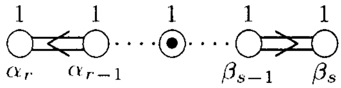
Here $\dim T_0 = 2l - 1 - l = l - 1$. If $l = 3$ we take $\Delta' = \{\alpha_3\}$, and by 3.5(i) codim $Z_{\Delta'} = 1$; so $\dim T_0^\lambda \leq 1$. Also, for $l > 3$ we take $\Delta' = \{\alpha_2, \ldots, \alpha_l\}$ and by 3.5(vi) codim $Z_{\Delta'} = (l - 1) - 1 = l - 2$; so again $\dim T_0^\lambda \leq 1$. Now by 2.5 we get $\dim \mathrm{Hom}_S(T \otimes_F T, T) = 1$.

- $D^{(2)}_{l+1}$ ($l \geq 2$) marked at one end



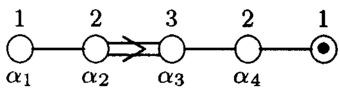
Since $\dim T_0 = \operatorname{rank} D_{l+1} - \operatorname{rank} B_l = 1$, the short roots are weights and it is enough to take $\Delta' = \{\alpha_l\}$ and apply 3.5(i) to conclude that $\operatorname{Hom}_S(T \otimes_F T, T) = 0$.

- $D_{l+1}^{(2)}$ $(l \geq 2)$ marked in the middle



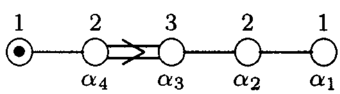
Here L is simple of type D_{l+1} $(D_3 = A_3)$ and S is a sum of two simple ideals of type B_r and B_s, respectively, with $r, s \geq 1$ $(B_1 = A_1)$ and $r + s = l$. Hence $\dim T_0 = 1$. If $r = s = 1$ we conclude by 2.5(ii) that $\dim \operatorname{Hom}_S(T \otimes_F T, T) = 1$. If, for instance, $r > 1$ we can take $\Delta' = \{\alpha_r\}$ and conclude from 3.5(i) that $\operatorname{Hom}_S(T \otimes_F T, T) = 0$.

- $E_6^{(2)}$ marked as



Then S is of type F_4, so that $\dim T_0 = 2$. With $\Delta' = \{\alpha_3\}$ (α_3 is a short root and hence a weight) we get $\operatorname{codim} Z_{\Delta'} = 1$ by 3.5(i); so by 2.5(iv) we get $\dim \operatorname{Hom}_S(T \otimes_F T, T) = 1$.

- Finally, $E_6^{(2)}$ marked as



With $\Delta' = \{\alpha_1, \alpha_2\}$ and because of 3.5(ii) we get $\operatorname{codim} Z_{\Delta'} = 2 = \dim T_0$; so $\operatorname{Hom}_S(T \otimes_F T, T) = 0$.

3.7

As a conclusion we get that in all cases $\dim \operatorname{Hom}_S(T \otimes_F T, T) = \dim T_0^{\lambda}$ and this is nonzero only in the cases contemplated in 3.2. So, summarizing all the work done, we get:

THEOREM. *Let T be a simple Lie triple system over an algebraically closed field F of characteristic zero and let $L = S \oplus T$ be its standard imbedding. Then* $\mathrm{Hom}_S(T \otimes_F T, T) = 0$ *unless either*:

(a) *T is of adjoint type other than A_n $(n \geq 2)$. In this case* $\mathrm{Hom}_S(T \otimes_F T, T) \cong \mathrm{Hom}_S(S \otimes_F S, S)$, *which is spanned by the Lie multiplication in S or*

(b) *there exists a simple Jordan algebra J of degree $n \geq 3$ such that T is the Lie triple system J_0. In this case either*:

(i) *J is not of type A (that is, J is not isomorphic to* $\mathrm{Mat}_n(F)^+$); *then* $\mathrm{Hom}_S(T \otimes_F T, T) = \mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes_F J_0, J_0)$ *is spanned by the product in* $(*)$.

(ii) *J is of type A; then J_0 is of adjoint type $S \cong sl(n, F)$ and* $\mathrm{Hom}_S(T \otimes_F T, T) = \mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes_F J_0, J_0)$ *is spanned by the product in* $(*)$ *and the Lie multiplication in $J_0 \cong sl(n, F)$.*

As mentioned in the Introduction, this theorem provides alternative proofs of [13, Theorem 8.1] and [14, Theorem 2.1].

4. ARBITRARY FIELDS

This section will be devoted to showing that the hypothesis in 3.7 that the field is algebraically closed can be avoided with only some minor modifications.

4.1

Given a Lie algebra L over a field K, two L-modules M and N and a subfield F of K, the F-descent of L is just L but considered as a Lie algebra over F. Then M and N are also modules for the F-descent of L. We can form $M \otimes_F N$, which is a module for the F-descent of L, and also $M \otimes_K N$, which is a module for L and, a fortiori, also a module for its F-descent.

We have then a natural homomorphisms of modules (for the F-descent of L)

$$p\colon M \otimes_F N \to M \otimes_K N$$
$$m \otimes n \mapsto m \otimes n$$

(with different meanings of the same sign $\otimes$).

LEMMA. *Let L be a Lie algebra over a field K, let M, N, and P be three irreducible nontrivial L-modules, and let F be a subfield of K such that K/F is a finite separable field extension. Then the linear map (of vector spaces* *over* K)

$$p^*: \operatorname{Hom}_L(M \otimes_K N, P) \to \operatorname{Hom}_L(M \otimes_F N, P)$$
$$\varphi \mapsto \varphi \circ p$$

(p *as above*) *is a bijection.*

Proof. Notice that $\operatorname{Hom}_L(M \otimes_F N, P)$ is a vector space over K because so is P. Let Ω be an algebraic closure of F and $\sigma_1, \ldots, \sigma_r: K \to \Omega$ the $r = [K : F]$ different F-homomorphisms. Then by separability

$$\Omega \otimes_F K = \Omega e_1 \oplus \cdots \oplus \Omega e_r$$

with $e_1, \ldots, e_r$ orthogonal idempotents. Besides, as a right K-vector space, $e_i \alpha = \sigma_i(\alpha) e_i$ for any $i = 1, \ldots, r$ and any $\alpha \in K$.

Now,

$$\begin{aligned} L_\Omega = \Omega \otimes_F L &\cong (\Omega \otimes_F K) \otimes_K L \\ &= (\Omega e_1 \otimes_K L) \oplus \cdots \oplus (\Omega e_r \otimes_K L) \\ &= L_1 \oplus \cdots \oplus L_r, \end{aligned}$$

where each $L_i = \Omega e_i \otimes_K L$ is the algebra obtained from L by extending scalars from K to Ω by means of σ_i. Similarly

$$M_\Omega = \Omega \otimes_F M \cong (\Omega e_1 \otimes_K M) \oplus \cdots \oplus (\Omega e_r \otimes_K M) = M_1 \oplus \cdots \oplus M_r,$$

and the same for N_Ω and P_Ω. By orthogonality, $L_i M_j = 0$ if $i \neq j$.

By scalar extension

$$\Omega \otimes_F \operatorname{Hom}_L(M \otimes_F N, P) \cong \operatorname{Hom}_{L_\Omega}(M_\Omega \otimes_\Omega N_\Omega, P_\Omega),$$

but

$$M_\Omega \otimes_\Omega N_\Omega = \bigoplus_{i,j=1}^{r} M_i \otimes_\Omega N_j.$$

For any $\phi \in \operatorname{Hom}_{L_\Omega}(M_i \otimes_\Omega N_j, P_k)$, if $i, j \neq k$

$$L_k \phi(M_i \otimes_\Omega N_j) \subseteq \phi(L_k M_i \otimes_\Omega N_j) + \phi(M_i \otimes_\Omega L_k N_j) = 0,$$

so that

$$\phi(M_i \otimes_\Omega N_j) = 0$$

since

$$\{x \in P_k : L_k x = 0\} = \Omega e_k \otimes \{x \in P : Lx = 0\} = 0;$$

while if $i = k \neq j$

$$\begin{aligned}\phi(M_i \otimes_\Omega N_j) &= \phi(M_i \otimes_\Omega L_j N_j) \\ &= \phi\big(L_j(M_i \otimes_\Omega N_j)\big) = L_j \phi(M_i \otimes_\Omega N_j) \subseteq L_j P_k = 0.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Hom}_{L_\Omega}(M_\Omega \otimes_\Omega N_\Omega, P_\Omega) &= \bigoplus_{i=1}^{r} \operatorname{Hom}_{L_i}(M_i \otimes_\Omega N_i, P_i) \\ &= \bigoplus_{i=1}^{r} \Omega e_i \otimes_K \operatorname{Hom}_L(M \otimes_K N, P).\end{aligned}$$

Hence

$$\begin{aligned}\dim_F \operatorname{Hom}_L(M \otimes_K N, P) &= r \dim_K \operatorname{Hom}_L(M \otimes_K N, P) \\ &= \sum_{i=1}^{r} \dim_\Omega \operatorname{Hom}_{L_i}(M_i \otimes_\Omega N_i, P_i) \\ &= \dim_\Omega \operatorname{Hom}_{L_\Omega}(M_\Omega \otimes_\Omega N_\Omega, P_\Omega) \\ &= \dim_F \operatorname{Hom}_L(M \otimes_F N, P).\end{aligned}$$

But $p\colon M \otimes_F N \to M \otimes_K N$ is onto; so p^* is one-to-one and hence, because of the same dimension, it is a bijection. ∎

As a consequence, if T is a simple Lie triple system over a field F of characteristic zero, Γ is its centroid and $S = [TT-] \subseteq gl(T)$, to determine $\operatorname{Hom}_S(T \otimes_F T, T)$ it is enough to determine $\operatorname{Hom}_S(T \otimes_\Gamma T, T)$. That is, it is enough to deal with central simple Lie triple systems. Moreover, for these systems we can extend scalars up to an algebraically closed field and then use Theorem 3.7.

4.2

Let T be a central simple Lie triple system over a field F of characteristic zero with standard imbedding $L = S \oplus T$, and let Ω be an algebraic closure of F. Assume there is a simple Jordan algebra of degree $n \geq 3$ over Ω such that $T_\Omega = \Omega \otimes_F T$ is the triple system J_0 constructed in 2.5. Then by Theorem 3.7, $\dim_F \operatorname{Hom}_S(S^2(T), T) = 1$, where $S^2(T)$ denotes the space of symmetric tensors in $T \otimes_F T$.

Let $\omega\colon T \times T \to T$ be symmetric, bilinear, and such that the map determined by $u \otimes u \mapsto \omega(u, u)$ $(u \in T)$ spans $\operatorname{Hom}_S(S^2(T), T)$. Then denoting by the same symbol ω its extension to a bilinear map $T_\Omega \times T_\Omega \to T_\Omega$, it follows that there is a $0 \neq \mu \in \Omega$ such that $x \cdot y = \mu\omega(x, y)$ for any $x, y \in T_\Omega = J_0$ ($x \cdot y$ as in $(*)$). Thus the multiplication in the Jordan algebra $J = \Omega 1 \oplus J_0$ is given by

$$(\alpha 1 + x) \circ (\beta 1 + y) = \left(\alpha\beta + \frac{1}{n} t(x, y)\right) 1 + (\alpha y + \beta x + x \cdot y), \quad (\diamondsuit)$$

where $t(x, y) = t(x \circ y)$ defines an invariant symmetric bilinear form on J_0. Since the algebra J is isomorphic (by means of the map $a \mapsto \mu a$) to the algebra defined on the same J but with the new multiplication given by $\mu^{-1} a \circ b$, we can assume that $x \cdot y = \omega(x, y)$ for any $x, y \in J_0$.

Now, the Jordan identity $(x^{\circ 2} \circ y) \circ x = x^{\circ 2} \circ (y \circ x)$ gives for $x, y \in J_0$ that

$$\frac{1}{n} t(x \cdot x, y) x + ((x \cdot x) \cdot y) \cdot x = \frac{1}{n} t(y, x) x \cdot x + (x \cdot x) \cdot (y \cdot x). \quad (\dagger)$$

On the other hand, $t \in \mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes_\Omega J_0, \Omega) \cong \Omega \otimes_F \mathrm{Hom}_S(T \otimes_F T, F)$, and by irreducibility of T and Schur's lemma the dimension of these spaces is 1; hence there are a $\lambda\colon T \times T \to F$ bilinear, symmetric, and nondegenerate and a $0 \neq \alpha \in \Omega$ such that $t(x, y) = \alpha\lambda(x, y)$ for any $x, y \in T$.

If $\alpha \notin F$, for any $x \in T$, take $y \in T$ with $\lambda(x, y) = 1$ and $(\dagger)$ allows us to conclude that $x \cdot x \in \Omega x$ for any $x \in T$. But this implies that the degree of J is at most 2, a contradiction. Hence $\alpha \in F$, $t(x, y) \in F$ for any $x, y \in T$ and $A = F1 \oplus T$ is closed under the Jordan product in J. In other words, $A = F1 \oplus T$ is a Jordan algebra over F which is a form of J. Moreover $S = \mathrm{Der}\, A$ since $S_\Omega = \mathrm{Der}\, J$. But *we cannot conclude that T is the Lie triple system associated with the simple Jordan algebra A* (because of the scaling we did of the product in J); we can only conclude that there is a nonzero scalar $\mu \in F$ such that $[xyz] = \mu((y \circ z) \circ x - y \circ (z \circ x))$ for any $x, y, z \in A_0 = T$. We will denote this triple system by (A_0, μ).

Let us put all this another way. For any central simple Jordan algebra J over F of degree ≥ 3 let $L(J) = \mathrm{Der}\, J \oplus J_0$ be the $\mathbb{Z}_2$-graded simple Lie algebra which is the standard imbedding of the Lie triple system J_0. Also, for any $\mathbb{Z}_2$-graded Lie algebra $L = S \oplus T$ and any $0 \neq \mu \in F$, let L_μ be the new $\mathbb{Z}_2$-graded Lie algebra constructed on the same vector space as L with the same grading and the same product of elements in S and of elements in S times elements in T, but with the new product of elements in T given by

$$[t_1, t_2]^\mu := \mu[t_1, t_2].$$

Then the arguments above prove the first part of:

PROPOSITION. *Let L be a $\mathbb{Z}_2$-graded simple Lie algebra over a field F and let Ω be an algebraic closure of F. Assume that there is a simple Jordan algebra J over Ω of degree ≥ 3 such that $L_\Omega \cong L(J)$. Then*:

(i) *There exist a nonzero scalar $\mu \in F$ and a Jordan algebra A over F, unique up to isomorphism, with $A_\Omega \cong J$ and $L \cong L(A)_\mu$ (as graded algebras).*

(ii) *$L(A)_\mu \cong L(A)_\nu$ (as graded algebras) if and only if $\mu^{-1}\nu \in F^2$.*

Proof. Assume that A and B are central simple Jordan algebras with $A_\Omega \cong J \cong B_\Omega$ and let $0 \neq \mu, \nu \in F$ and ϕ: $L(A)_\mu \to L(B)_\nu$ an isomorphism of graded algebras. Then it is immediate to check that ϕ is an isomorphism from $L(A)$ to $L(B)_{\mu^{-1}\nu}$, so that we may assume $\mu = 1$. Let $S = \operatorname{Der} A$, then the restriction $\phi|_S$: $\operatorname{Der} A \to \operatorname{Der} B$ is an isomorphism which makes B_0 an irreducible S-module. Let $\varphi = \phi|_{A_0}$: $A_0 \to B_0$, then since $\operatorname{Hom}_S(S^2(A_0), A_0)$ and $\operatorname{Hom}_S(S^2(B_0), B_0) = \operatorname{Hom}_{\operatorname{Der} B}(S^2(B_0), B_0)$ have dimension 1, there is a nonzero scalar $0 \neq \eta \in F$ such that $\varphi(a_1 \cdot a_2) = \eta\varphi(a_1) \cdot \varphi(a_2)$ for any $a_1, a_2 \in A_0$. Therefore $(\eta\varphi)(a_1 \cdot a_2) = (\eta\varphi)(a_1) \cdot (\eta\varphi)(a_2)$ and $\eta\varphi$ is an isomorphism between the algebras $(A_0, \cdot)$ and $(B_0, \cdot)$. By (†) and since the degree of the Jordan algebras A and B is ≥ 3 it follows that the traces of A and B are determined by the algebras $(A_0, \cdot)$ and $(B_0, \cdot)$ respectively. From the expression in (◇) $\eta\varphi$ extends to an isomorphism of Jordan algebras ψ: $A \to B$ (with $\psi(1) = 1$ and $\psi(a) = \eta\varphi(a)$ for any $a \in A_0$). This shows the uniqueness in (i). Besides, ψ induces an isomorphism Ψ: $L(A) \to L(B)$ by means of $\Psi(s) = \psi \circ s \circ \psi^{-1}$ for any $s \in S$ and $\Psi(a) = \psi(a)$ for any $a \in A_0$. Now, $\phi \circ \Psi^{-1}$: $L(B) \to L(B)_\nu$ is an isomorphism of simple graded Lie algebras with $\phi \circ \Psi^{-1}(b) = \eta^{-1}b$ for any $b \in B_0$ (the odd part). But for any $b_1, b_2 \in B_0$

$$\phi \circ \Psi^{-1}([b_1, b_2]) = \left[\phi \circ \psi^{-1}(b_1), \phi \circ \psi^{-1}(b_2)\right]^\nu = \eta^{-2}\nu[b_1, b_2]$$

and for any $b_1, b_2, b_3 \in B_0$

$$\begin{aligned} \eta^{-1}[[b_1, b_2], b_3] &= \phi \circ \psi^{-1}([[b_1, b_2], b_3]) \\ &= \left[\phi \circ \Psi^{-1}([b_1, b_2]),\ \phi \circ \psi^{-1}(b_3)\right] \\ &= \eta^{-2}\nu\eta^{-1}[[b_1, b_2], b_3]. \end{aligned}$$

Thus $\eta^{-2}\nu = 1$, giving the converse in (ii).

Conversely, if $\mu^{-1}\nu = \eta^2$ for $0 \neq \eta \in F$, the map ϕ: $L(A)_\nu \to L(A)_\mu$ given by $\phi(s) = s$ for any $s \in S$ and $\phi(t) = \eta t$ for any $t \in A_0$ is easily seen to be an isomorphism. ∎

Remarks. (i) The proposition above can be deduced also from [18, Theorem 4.5]. The proof above is more in the spirit of the rest of the paper.

(ii) Over the real field, attached to any central simple Jordan algebra A of degree ≥ 3 there appear two nonisomorphic triple systems with standard imbeddings $L(A)$ and $L(A)_{-1}$. One is the dual of the other [7, p. 235]. The irreducible compact simply connected Riemannian symmetric spaces of type I with noncanonical affine connections (see [14, Theorem 2.1]) appear as the duals of the ones associated with the Jordan algebras of symmetric real matrices, symmetric matrices of quaternions with respect to the standard involution, and the simple exceptional Jordan algebra $H_3(\mathbb{O})$.

4.3

To state the result that summarizes most of the work done throughout the paper, we need one more thing. Let J be a central simple Jordan algebra of type A (that is, after extension of scalars it becomes isomorphic to $\mathrm{Mat}_n(K)^+$) over a field Γ of characteristic zero. Then (see [9, Chaps. IV and V]) either

(i) there is a central simple associative algebra B over Γ such that J is the Jordan algebra B^+ or

(ii) there is a central simple associative algebra A over a quadratic field extension $P = \Gamma[q]$ $(q^2 \in \Gamma)$ equipped with an involution of second kind j such that J is the Jordan algebra of symmetric elements $H(A, j) = \{x \in A : j(x) = x\}$.

In the first case put $P = \Gamma \oplus \Gamma = \Gamma[q]$ with $q^2 = 1$, $A = B \oplus B^{\mathrm{op}}$ (where op denotes the opposite algebra), and j the involution of the second kind given by $j(b_1, b_2) = (b_2, b_1)$. So in both cases $J = H(A, j)$ for a central simple involutorial algebra (A, j) of the second kind.

Then, $\mathrm{Der}\, J = S(A, j)_0 = [S(A, j), S(A, j)]$ acting on $H(A, j)$ by means of the Lie bracket in A, where $S(A, j) = \{x \in A : j(x) = -x\}$ (see [9, Theorem 6.9]). Besides, $H(A, j) = qS(A, j)$, $H(A, j)_0$ is the adjoint module for $S(A, j)_0$, and under the isomorphism of $S(A, j)_0$-modules given by $H(A, j)_0 \to S(A, j)_0$, $x \mapsto qx$, the Lie bracket on $S(A, j)_0$ becomes the $S(A, j)_0$-invariant product on $H(A, j)_0$

$$x \star y = q(xy - yx). \qquad (**)$$

Now, Theorem 3.7 plus the arguments in 4.1 and 4.2 prove the following:

THEOREM. *Let T be a simple Lie triple system over a field F of characteristic zero with centroid Γ and let $L = S \oplus T$ be its standard imbedding. Then* $\mathrm{Hom}_S(T \otimes_F T, T) = 0$ *unless either*:

(a) *T is of adjoint type with S being a central simple Lie algebra over Γ of type different from A_n $(n \geq 2)$. In this case* $\mathrm{Hom}_S(T \otimes_F T, T) \cong \mathrm{Hom}_S(S \otimes_F S, S)$, *which is spanned over Γ by the Lie multiplication in S.*

(b) *There exists a central simple Jordan algebra* J *of degree* $n \geq 3$ *over* Γ *and a nonzero scalar* $\mu \in \Gamma$ *(which can be taken modulo* Γ^2*) such that* T *is the Lie triple system* (J_0, μ). *In this case either*:

(i) J *is not of type* A *over* Γ; *then* $\mathrm{Hom}_S(T \otimes_F T, T) = \mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes_F J_0, J_0)$ *is spanned over* Γ *by the product in* $(*)$.

(ii) $J = H(A, j)$ *for some central simple associative involutorial algebra* (A, j) *of the second kind over* Γ. *Let* $P = \Gamma[q]$ $(0 \neq q^2 \in \Gamma)$ *be the center of* A. *Then*

$$\begin{aligned}\mathrm{Hom}_S(T \otimes_F T, T) &= \mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes_F J_0, J_0)\\ &= \mathrm{Hom}_{S(A,j)_0}\big(H(A,j)_0 \otimes_\Gamma H(A,j)_0, H(A,j)_0\big)\end{aligned}$$

is spanned over Γ *by the products in* $(*)$ *and* $(**)$.

This theorem may be considered as an extension of the classification of the flexible Lie-admissible algebras over a field of characteristic zero with an underlying simple Lie algebra (see [17, Chap. III] and the references therein), since this classification is subsumed in cases (a) and (b)(ii) of the above result.

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