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The Lattice of Ideals of a Lie Algebra*, †

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In this paper the Lie algebras in which the lattice formed by the ideals is complemented or complemented and distributive are classified. Moreover, it is shown that the derived algebra (arbitrary characteristic) and the solvable radical (characteristic zero) can be characterized in terms of the ideal lattice structure. The relationship between Lie algebras having isomorphic lattices of ideals is also studied. It turns out that, over algebraically closed fields of characteristic zero, the Frattini ideal is preserved under ideal lattice isomorphisms and, as a consequence of this fact, the nilpotent radical is preserved by this kind of isomorphism when the codimension of the derived algebra is at least two.

1. INTRODUCTION

Let L and M be finite dimensional Lie algebras over the field F. The ideals of L may be taken as the elements of a lattice under the operations of sum and intersection. We denote by $\Im(L)$ the lattice of all ideals of L. By a lattice isomorphism from $\Im(L)$ onto $\Im(M)$ we mean a bijective map $\alpha\colon \Im(L) \to \Im(M)$ such that

$$\alpha(I \cap J) = \alpha(I) \cap \alpha(J) \qquad \text{and} \qquad \alpha(I + J) = \alpha(I) + \alpha(J).$$

We are interested in the following problem: What is the relation between the structure of a Lie algebra and that of its lattice of ideals? In the present paper, we shall tackle this problem from two different points of view. The first one is the characterization of a Lie algebra (or a class) through its lattice of ideals and the second is the study of the relationship between Lie algebras whose lattices of ideals are isomorphic. Analogous

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questions for the lattice of subalgebras have been considered by several authors; a current summary can be found in [6].

In Section 2, we classify the Lie algebras in which the lattice formed by the ideals is complemented or complemented and distributive or abelian; the corresponding problem for the subalgebra lattice has been studied in [8, 13]. As a corollary we show that in the case dim $L/L^2 > 1$, the derived algebra (for arbitrary fields) and the solvable radical of L (for fields of characteristic zero) can be characterized in terms of the ideal lattice structure of L. These results will be used in Section 3, where we investigate lattice isomorphisms. In this section, we consider the class of the Lie algebras having the property $\mathfrak{P}$: "The intersection of all maximal subalgebras is trivial" (i.e., the Frattini subalgebra is zero). When F has characteristic zero, we show that if $\mathfrak{P}$ holds in L and M is such that $\mathfrak{I}(L)$ and $\mathfrak{I}(M)$ are isomorphic, either $\mathfrak{P}$ holds in M or F is not algebraically closed and L and M both have a severely constrained structure. Since any Lie algebra over a field of characteristic zero can be mapped onto a Lie algebra in which $\mathfrak{P}$ holds, the results in this section provide information about the general problem of lattice isomorphisms. The final section gives counter-examples in positive characteristic for some of the results obtained in Section 3 for Lie algebras in characteristic zero.

Every Lie algebra considered in this paper will be finite dimensional over a field F. Rad(L) (resp. Nil(L)) denotes the largest solvable (resp. nilpotent) ideal of L. The Jacobson radical, $J(L)$, is the intersection of all maximal ideals of L. We denote the terms of the lower central series of L by $L = L^1$ and $L^i = [L, L^{i-1}]$ for $i > 1$. The center of L is denoted by $Z(L)$. We shall say that L has nilpotency index n if $L^n = 0$ but $L^{n-1} \neq 0$. We denote by $L^\infty = \bigcap_{n=1}^{\infty} L^n$. $\mathfrak{I}(L)$ denotes the lattice of all ideals of L. For $P, Q \in \mathfrak{I}(L)$, we define the interval $[P:Q] = \{R \in \mathfrak{I}(L): P \leq R \leq Q\}$. Clearly, $[P:Q]$ is a sublattice of $\mathfrak{I}(L)$. If $\mathfrak{C} = \{P_i\}_{0 \leq i \leq n}$ is a subset of $\mathfrak{I}(L)$ such that $P_i < P_{i+1}$, we shall say that $\mathfrak{C}$ is an $(n+1)$-element chain of length n (see [3, p. 2]). The length of the largest chain in $\mathfrak{I}(L)$ will be called the length of $\mathfrak{I}(L)$. We define the socle of L, Soc(L), to be the sum of all minimal ideals of L. We also define the abelian socle (or zerosocle) of L, Asoc(L), to be the sum of all minimal abelian ideals. For every x in L we denote by ad x the right multiplication, $C_L(x) = \{b \in L: [b, x] = 0\}$ and $\mathrm{ad}_A B = \{\mathrm{ad}\, b$ restricted to A; $b \in B\}$. Algebra direct sums are denoted by $\oplus$, whereas direct sums of vector space structures are denoted by $\dotplus$.

2. Complemented Ideal Lattices

We start with a list of definitions of terms in general Lattice Theory (see [3]) and a preliminary lemma that was obtained by Ore in [9].

Let $(\mathfrak{L}, \vee, \wedge)$ be a lattice. We say that $\mathfrak{L}$ has *unit* element if there exists $1 \in \mathfrak{L}$ such that $a = a \wedge 1$ for all $a \in \mathfrak{L}$. Dually, $\mathfrak{L}$ is said to have a *zero* element if there exists $0 \in \mathfrak{L}$ such that $a = a \vee 0$ for all $a \in \mathfrak{L}$. The lattice $\mathfrak{L}$ is called *bounded* if it has 0 and 1. If $\mathfrak{L}$ is of finite length, an element $a \in \mathfrak{L}$ is called an *atom* if a is a minimal element in the set of nonzero elements. Dually, an element $b \in \mathfrak{L}$ is called a *co-atom* if b is a maximal element in the set of non-unit elements. We define the *Jacobson radical*, $J(\mathfrak{L})$, of $\mathfrak{L}$ to be the intersection of all co-atoms of $\mathfrak{L}$. Note that if L is a Lie algebra, $\mathfrak{I}(L)$ is a bounded lattice in which the atoms are the minimal nonzero ideals of L and the co-atoms the maximal ones. We also observe that the Jacobson radical, $J(\mathfrak{I}(L))$, of $\mathfrak{I}(L)$ coincides with the Jacobson radical, $J(L)$, of L.

2.1. Definition. Let $(\mathfrak{L}, \vee, \wedge)$ be a bounded lattice. By a complement of an element $P \in \mathfrak{L}$ is meant an element $Q \in \mathfrak{L}$ such that $P \vee Q = 1$ and $P \wedge Q = 0$.

2.2. Definition. Let $(\mathfrak{L}, \vee, \wedge)$ be a bounded lattice. Then:

(1) $\mathfrak{L}$ is called complemented if all its elements have complements.

(2) $\mathfrak{L}$ is called abelian if $\mathfrak{L}$ is complemented and given P, Q atoms of $\mathfrak{L}$ an atom C exists, different from P and Q, such that $P \vee Q = C \vee Q = C \vee P$.

(3) $\mathfrak{L}$ is called distributive if the following identity holds in $\mathfrak{L}$:

$$(P \vee Q) \wedge (Q \vee R) \wedge (R \vee P) = (P \wedge Q) \vee (Q \wedge R) \vee (R \wedge P). \tag{2.2.3}$$

2.3. Lemma [9]. *Let L be a Lie algebra over a field of any characteristic and P, Q, R be ideals of L. The following two identities hold*:

(1) $(P + Q) \cap (Q + R) \cap (R + P) = P \cap (Q + R) + Q \cap (R + P) + Q \cap R + R \cap P$.

(2) $(P \cap Q) + (Q \cap R) + (R \cap P) = (P + (Q \cap R)) \cap (Q + R) \cap (Q + (R \cap P)) \cap (R + P)$.

2.4. Theorem. *Let L be a Lie algebra over a field F of any characteristic. Then, we have the following*:

(i) *$\mathfrak{I}(L)$ is a complemented lattice if and only if $L = A \oplus S$, where A is an abelian ideal and S is a direct sum of ideals of L which are simple Lie algebras.*

(ii) *$\mathfrak{I}(L)$ is an abelian lattice of length at least 2 if and only if L is abelian of dimension at least 2. In that case, the dimension of L is equal to the length of $\mathfrak{I}(L)$.*

(iii) *$\Im(L)$ is a complemented distributive lattice if and only if L is of the type described in* (i) *with A of dimension at most one.*

Proof. (i) Suppose first that $\Im(L)$ is complemented. Let Q be a complement of $\text{Asoc}(L)$. Then, $L = \text{Asoc}(L) \oplus Q$ and $\text{Rad}(L) = \text{Asoc}(L) \oplus (\text{Rad}(L) \cap Q)$. We conclude that there are no minimal abelian ideals contained in $\text{Rad}(L) \cap Q$, thus $\text{Rad}(L) = \text{Asoc}(L)$. Moreover, we can decompose $Q = S_1 \oplus \cdots \oplus S_m$, where each S_i is a simple ideal which proves the result. To prove the converse, consider the set of linear transformations $\text{ad}\, L = \{\text{ad}\, a : a \in L\}$. The invariant subspaces relative to this set are the ideals of L. Since $L = L_1 \oplus \cdots \oplus L_n$, where each L_i is a minimal ideal, we see that the set $\text{ad}\, L$ is completely reducible on L and therefore $\Im(L)$ is complemented.

(ii) Assume $\Im(L)$ is abelian and let N be a simple minimal ideal of L. As $\Im(L)$ is complemented of length at least 2, we can take K as a minimal ideal of L different from N. The minimality of N implies $N \cap K = 0$. Now, by the second condition of the abelian lattice definition, there exists a minimal ideal C such that $N \oplus C = K \oplus C = N \oplus K$. Then, N, K, and C are isomorphic Lie algebras and therefore K and C are simple ideals. On the other hand, we have $[C, N] \leq C \cap N = 0$ and, in an analogous way, $[C, K] = 0$. Thus $[C, C] \leq [C, N \oplus K] = 0$, a contradiction. We conclude that all the minimal ideals of L are abelian.

Next denote by S the sum of all minimal ideals of L. By the preceding paragraph, S is abelian. From the fact that $\Im(L)$ is complemented it follows that $S = L$. It is clear that the dimension of L coincides with the length of $\Im(L)$. The converse is immediate.

(iii) Suppose first that $\Im(L)$ is complemented and distributive. It follows that $L = A \oplus S$ with A and S as in (i). Assume $\dim A > 1$ and take $a, b \in A$ such that $\{a, b\}$ is a linearly independent set. Then, $F(a + b)$ is an ideal of L satisfying

$$F(a+b) \oplus F(a) = F(a+b) \oplus F(b) = F(a) \oplus F(b).$$

Thus the distributive law (2.2.3) fails for the ideals $F(a + b)$, $F(a)$, $F(b)$, a contradiction. It follows that the dimension of A is at most one.

Now we shall prove the converse. For P, Q, R ideals of L, we need prove that (2.2.3) holds, i.e., $(P + Q) \cap (Q + R) \cap (R + P) = (P \cap Q) + (Q \cap R) + (R \cap P)$. Define

$$T(P) = (P + (Q \cap R)) \cap (Q + R),$$
$$T(Q) = (Q + (R \cap P)) \cap (R + P),$$

and

$$T(R) = (R + (P \cap Q)) \cap (P + Q).$$

From Lemma 2.3 we have the identities

$$(P + Q) \cap (Q + R) \cap (R + P) = T(P) + T(Q)$$
$$= T(Q) + T(R) = T(R) + T(P) = U$$
$$(P \cap Q) + (Q \cap R) + (R \cap P) = T(P) \cap T(Q)$$
$$= T(Q) \cap T(R) = T(R) \cap T(P) = V.$$

Note that (2.2.3) holds if and only if $U \leq V$. From the above identities, we obtain $(T(P)/V) \oplus (T(Q)/V) = (T(Q)/V) \oplus (T(R)/V) = (T(R)/V) \oplus (T(P)/V) = U/V$ and therefore $T(P)/V$ and $T(Q)/V$ are contained in $Z(U/V)$, which implies $U^2 \leq V$. The case $U = 0$ is immediate. If U is a semisimple ideal of L, it follows that $U = S_1 \oplus \cdots \oplus S_r$, where each S_i is a simple ideal, so $U^2 = U \leq V$. Otherwise $0 \neq A = F(z)$ and $U = F(z) \oplus S$, where S is a semisimple ideal of L. In that case, $U^2 = S \leq V$. Moreover, it is easy to check that $F(z) \leq U$ implies:

(a) $F(z) \leq P$ or $F(z) \leq Q$,

(b) $F(z) \leq Q$ or $F(z) \leq R$,

(c) $F(z) \leq R$ or $F(z) \leq P$.

The inequalities (a), (b), and (c) yield $F(z) \leq V$ and so $U \leq V$. Hence (2.2.3) holds. ∎

The result established above reveals that every n-dimensional abelian Lie algebra is determined by its lattice of ideals unless $n = 1$. But the ideal lattice structure does not always determine the structure of a Lie algebra uniquely. Clearly, the one-dimensional Lie algebra and every simple Lie algebra have the same lattice of ideals consisting merely of a two-element chain. This example shows that there exist non-isomorphic Lie algebras which have isomorphic lattices of ideals and, in addition, the solvable radical and the derived algebra are not preserved in general by ideal lattice isomorphisms. However, by imposing some restrictions which are connected with the abelian lattices, it is possible to characterize the derived algebra and the solvable radical by means of the ideal lattice structure. Our following task will be to show this and to determine which of the Lie algebras described in (i) and (iii) of Theorem 2.4 have isomorphic lattices.

Let L be a Lie algebra and consider the Jacobson radical, $J(L)$, of L and the Jacobson radical, $J(\mathfrak{I}(L))$, of $\mathfrak{I}(L)$. From the first paragraph of this section we have $J(L) = J(\mathfrak{I}(L))$. Assume L is taken over a field is of characteristic zero. It is known that $J(L) = L^2 \cap \mathrm{Rad}(L) =$

$[\mathrm{Rad}(L), L] \leq \mathrm{Nil}(L)$ (see [4, p. 91; 7, Theorem 1]). Next, let S be a semisimple Levi factor of L and let M be the Lie algebra $L/J(L)$. Clearly, $\mathrm{Rad}(M) = \mathrm{Rad}(L)/J(L) = \mathrm{Rad}(L)/[\mathrm{Rad}(L), L]$ is abelian and $S_1 = S + J(L)/J(L)$ is a semisimple Levi factor of M such that $[S_1, M] = S_1$. Then S_1 is an ideal of M and, since the base field is of characteristic zero, the direct sum of simple ideals. It follows that (i) of Theorem 2.4 holds in $L/J(L)$ and therefore $\Im(L/J(L))$ is a complemented lattice.

2.5. Corollary. *Let L be a Lie algebra over an arbitrary field F. We have the following*:

(i) *If L has an ideal K such that the interval $[K : L]$ is an abelian lattice of length at least* 2, *then* $\dim L/L^2 > 1$. *In this case, L^2 is equal to the intersection of all $K \in \Im(L)$ for which the interval $[K : L]$ is an abelian lattice of length at least* 2.

(ii) *If L has an ideal N such that the interval $[0 : N]$ is abelian of length at least* 2, *then N is an abelian ideal and it is contained in* $\mathrm{Asoc}(L)$.

(iii) *If the characteristic of F is zero and L has an ideal K such that the interval $[J(\Im(L)) : K]$ is an abelian lattice of length at least* 2, *then* $\dim \mathrm{Rad}(L)/J(L) > 1$. *In this case,* $\mathrm{Rad}(L)$ *is equal to the sum of all elements $K \in \Im(L)$ for which the interval $[J(\Im(L)) : K]$ is an abelian lattice of length at least* 2.

Proof. (i) Consider L/K and apply (ii) in Theorem 2.4.

(ii) Let K be a minimal ideal contained in N. As $[0 : N]$ is complemented of length at least 2, there exists a minimal ideal P different from K contained in N. As in the proof of (ii) in Theorem 2.4, we conclude that every minimal ideal of L contained in N is abelian. Now, from the definition of a complemented lattice it follows that N is contained in $\mathrm{Asoc}(L)$.

(iii) Denote $L/J(L)$ by M and observe that $\Im(M)$ is a complemented and not a distributive lattice. Then, (i) and (iii) of Theorem 2.4 imply $Z(M) = \mathrm{Rad}(M)$ and $\dim Z(M) \geq 2$. As $\mathrm{Rad}(M) = \mathrm{Rad}(L)/J(L)$, we conclude $\dim \mathrm{Rad}(L)/J(L) > 1$ and the interval $[J(\Im(L)) : \mathrm{Rad}(L)]$ is abelian of length at least 2. Now, let N be an ideal such that $[J(\Im(L)) : N]$ is abelian of length at least 2. From (ii), we get $N/J(L)$ is abelian. Since $J(L)$ is nilpotent, we obtain that N is a solvable ideal and therefore $N \leq \mathrm{Rad}(L)$. ∎

Moreover, we have the following characterization for solvable Lie algebras of characteristic zero in terms of their ideal lattice:

2.6. Corollary. *Let L be a Lie algebra over a field of characteristic zero such that $\Im(L)$ has at least two co-atoms. Then, the interval*

$[J(\Im(L)) : L]$ is abelian if and only if L is solvable. In this case, the dimension of L/L^2 is equal to the length of the interval $[J(\Im(L)) : L]$.

Proof. The "only if" follows from (iii) of Corollary 2.5. Now, suppose L is solvable. Then, $L^2 = J(L) = J(\Im(L))$ and therefore the result follows. ∎

The following example shows that (iii) of Corollary 2.5 and Corollary 2.6 fail if L is taken over a field of positive characteristic:

2.7. Example. Let F be a field of characteristic $p > 0$. Consider the Lie algebra $gl(V)$ of linear transformations on an n-dimensional vector space V, where n is divisible by p and $n > 2$. The only proper ideals in $gl(V)$ are $gl(V)^2$ and FI, the set of multiples of the identity. Since $gl(V)^2$ is the set of linear transformations of trace zero and $\operatorname{tr} I = n = 0$, $FI \leq gl(V)^2$. Hence $gl(V)/FI$ has only one proper ideal, which is $gl(V)^2/FI$ and this ideal is simple. Denote by $M = gl(V)/FI$ and let $F(a)$ be a one-dimensional Lie algebra. Define L as the direct sum of the vector spaces M and $F(a)$ and the Lie bracket by declaring that $[a, m] = 0$ for every $m \in M$ and M keeps its original Lie bracket. Then, $L = M \oplus F(a)$ is a Lie algebra satisfying $\operatorname{Rad}(L) = F(a)$, $J(\Im(L)) = L^2 = M^2$, and $[L^2 : L]$ is abelian of length 2.

2.8. Corollary. *Let L and M be Lie algebras over a field F of any characteristic such that L is of the type described in* (i) *of Theorem* 2.4. *Then, $\Im(L)$ and $\Im(M)$ are isomorphic if and only if M is as in* (i) *of Theorem* 2.4 *and one of the following holds*:

(i) $\dim Z(L) = \dim Z(M)$ *and the numbers of simple ideals of L and M coincide.*

(ii) $\dim Z(L) = 1$, $Z(M) = 0$, *and M has one simple ideal more than L.*

(iii) $Z(L) = 0$, $\dim Z(M) = 1$, *and L has one simple ideal more than M.*

Proof. Suppose first that $\Im(L)$ and $\Im(M)$ are isomorphic. From (i) of Theorem 2.4, we can write $L = Z(L) \oplus L_1 \oplus \cdots \oplus L_s$ and $M = Z(M) \oplus M_1 \oplus \cdots \oplus M_r$, where L_i and M_j are simple ideals. If $\Im(L)$ is distributive, it is easy to prove the result from (iii) of Theorem 2.4. Assume then that $\Im(L)$ is not distributive. In that case, $\dim Z(L) \geq 2$ and, given α an isomorphism from $\Im(L)$ onto $\Im(M)$, (ii) of Corollary 2.5 implies

$$\alpha(Z(L)) \leq \operatorname{Asoc}(M) = Z(M) \quad \text{and}$$
$$\alpha^{-1}(Z(M)) \leq \operatorname{Asoc}(L) = Z(L).$$

Therefore, $\alpha(Z(L)) = Z(M)$, which proves the result because $\dim Z(L)$ coincides with the length of the interval $[0 : Z(L)]$.

Now we shall prove the converse. We can decompose L and M as in the first paragraph. Note that, if P (resp. N) is an ideal of L (resp. M), $P = (P \cap Z(L)) \oplus K$ (resp. $N = (N \cap Z(M)) \oplus R$), where K (resp. R) is of the form $L_{i_1} \oplus \cdots \oplus L_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq s$ (resp. $M_{i_1} \oplus \cdots \oplus M_{i_q}$ $1 \leq i_1 < \cdots < i_q \leq r$). If $\dim Z(L) \leq 1$, the result is easily verified. Suppose then that $\dim Z(L) > 1$. In that case, (i) holds in L and M and therefore $r = s$ and $Z(L), Z(M)$ are F-isomorphic vector spaces. Consequently, there exists a lattice isomorphism α from $[0 : Z(L)]$ onto $[0 : Z(M)]$. On the other hand, if we denote by $\tilde{L} = L_1 \oplus \cdots \oplus L_s$ and $\tilde{M} = M_1 \oplus \cdots \oplus M_s$, it is immediate that β: $[0 : \tilde{L}] \to [0 : \tilde{M}]$ defined by

$$\beta(0) := 0 \qquad \text{and} \qquad \beta\left(L_{i_1} \oplus \cdots \oplus L_{i_p}\right) := M_{i_1} \oplus \cdots \oplus M_{i_p}$$

is a lattice isomorphism. Next let γ: $\Im(L) \to \Im(M)$ be defined as follows: $\gamma(P) := \alpha(P \cap Z(L)) \oplus \beta(L_{i_1} \oplus \cdots \oplus L_{i_p})$ for every $P \in \Im(L)$. It is easily checked that γ is a lattice isomorphism. ▌

2.9. COROLLARY. *Let L and M be Lie algebras over a field F and α: $\Im(L) \to \Im(M)$ be a lattice isomorphism. We have the following*:

(1) *If* $\dim L/L^2 > 1$, *then*:

(i) $\alpha(L^2) = M^2$ *and* $\dim L/L^2 = \dim M/M^2$.

In addition, if the characteristic of F is zero we have the following:

(ii) $\alpha(\mathrm{Rad}(L)) = \mathrm{Rad}(M)$.

(iii) *The number of simple ideals of the semisimple Levi factors of L and M coincides.*

(iv) $\alpha(\mathrm{Asoc}(L)) = \mathrm{Asoc}(M)$ *and the number of simple ideals of L and M coincides.*

(2) *If* $\dim Z(L) > 1$, *then* $\alpha(Z(L)) \leq \mathrm{Asoc}(M)$.

Proof. (1.i) Apply (i) of Corollary 2.5 and (ii) of Theorem 2.4.

(1.ii) Since $J(L) = L^2 \cap \mathrm{Rad}(L)$, we obtain $L/L^2 = L^2 + \mathrm{Rad}(L)/L^2 \cong \mathrm{Rad}(L)/J(L)$. Then the result follows from (iii) of Corollary 2.5.

(1.iii) From (1.ii), the Lie algebras $L/\mathrm{Rad}(L)$ and $M/\mathrm{Rad}(M)$ have isomorphic lattices of ideals. Thus the result follows from (i) of Theorem 2.4, Corollary 2.8, and the Malcev–Harish-Chandra Theorem.

(1.iv) It is clear that $\mathrm{Asoc}(L) = \mathrm{Rad}(L) \cap \mathrm{Soc}(L)$. Then from (1.ii):

$$\begin{aligned}\alpha(\mathrm{Asoc}(L)) &= \alpha(\mathrm{Rad}(L)) \cap \alpha(\mathrm{Soc}(L)) \\ &= \mathrm{Rad}(M) \cap \mathrm{Soc}(M) = \mathrm{Asoc}(M).\end{aligned}$$

On the other hand, the sum of all simple ideals of L is equal to $\mathrm{Soc}(L)^2$ and therefore, $\mathrm{Soc}(L) = \mathrm{Asoc}(L) \oplus \mathrm{Soc}(L)^2$. For every simple ideal P of L we have that $P \cap \mathrm{Asoc}(L) = 0$. Then, $\alpha(P) \cap \mathrm{Asoc}(M) = 0$, which implies $\alpha(P) \leq \mathrm{Soc}(M)^2$ and therefore $\alpha(\mathrm{Soc}(L)^2) = \mathrm{Soc}(M)^2$. Since the number of simple ideals of L coincides with the length of $[0 : \mathrm{Soc}(L)^2]$, the last assertion is immediate.

(2) Apply (ii) of Corollary 2.5. ∎

The following example shows that (1.ii) and (1.iv) of Corollary 2.9 fail if L is taken over a field of positive characteristic:

2.10. Example. Let F be a field of characteristic $p > 0$ and $L = M \oplus F(a)$ be as in Example 2.7. Note that $M = M^2 \dotplus F(x)$ and then we can write $L = M^2 \dotplus F(x, a)$. We have that $F(a)$ and M^2 are the only minimal ideals of L. If P is a proper ideal of L which is not minimal, we get that P is maximal because the length of $\mathfrak{I}(L)$ is 3, thus $L^2 \leq P$. It follows that the ideals of L are 0, L, $F(a)$, $L^2 = M^2$, $L^2 \oplus F(a)$, and $L^2 \dotplus F(x + \lambda a)$ with $\lambda \in F$.

Next consider the 3-dimensional Lie algebra K with basis $\{b, c, z\}$ and products $[b, z] = [c, z] = 0$, $[b, c] = b$. It is easy to check that the ideals of K are $0, K, F(z), K^2 = F(b)$, $F(b, z)$, and $F(b, c + \lambda z)$ with $\lambda \in F$. Now, define α: $\mathfrak{I}(L) \to \mathfrak{I}(K)$ in the following way:

$$\alpha(0) := 0, \qquad \alpha(L) := K, \qquad \alpha(F(a)) := F(z), \qquad \alpha(L^2) := F(b),$$

and

$$\alpha\big(L^2 \dotplus F(\mu x + \lambda a)\big) := F(b, \mu c + \lambda z) \qquad \text{for } (\mu, \lambda) \neq (0, 0).$$

From Table I, it is easily checked that α is a lattice isomorphism such that

$$\alpha(\mathrm{Rad}(L)) = F(z) \neq \mathrm{Rad}(K) = K \qquad \text{and}$$
$$\alpha(\mathrm{Asoc}(L)) = F(z) \neq \mathrm{Asoc}(K) = F(b, z).$$

3. The Lattice of Ideals of a φ-Free Lie Algebra

Let L be a Lie algebra over an arbitrary field F. The intersection of all maximal subalgebras of L, $F(L)$, is called the Frattini subalgebra of L. If L is a solvable Lie algebra Barnes and Gastineau-Hills [1, Lemma 3.4] proved that $F(L)$ is an ideal. This statement, which fails in the general case, is true for any Lie algebra if the underlying field has characteristic

TABLE I

α(Q) \ α(P)		$K^2 + F(c + \mu z)$	$K^2 \oplus F(z)$	K^2
$K^2 + F(c + \delta z)$, $\mu \neq \delta$	+	K		
	∩	K^2		
$K^2 \oplus F(z)$	+	K		
	∩	K^2		
K^2	+	$K^2 + F(c + \mu z)$	$K^2 \oplus F(z)$	
	∩	K^2	K^2	
$F(z)$	+	K	$K^2 \oplus F(z)$	$K^2 \oplus F(z)$
	∩	0	$F(z)$	0

+/∩	L^2	$L^2 \oplus F(a)$	$L^2 + F(x + \mu a)$	P \ Q
+	$L^2 \oplus F(a)$	$L^2 \oplus F(a)$	L	$F(a)$
∩	0	$F(a)$	0	
+		$L^2 \oplus F(a)$	$L^2 + F(x + \mu a)$	L^2
∩		L^2	L^2	
+			L	$L^2 \oplus F(a)$
∩			L^2	
+			L	$L^2 + F(x + \delta a)$, $\mu \neq \delta$
∩			L^2	

zero (see [12, Corollary 3.3]). This permits us to define the Frattini ideal, $\varphi(L)$, of L as the largest ideal of L contained inside $F(L)$. Following [12], L is said to be φ-free if $\varphi(L) = 0$. Note that for all Lie algebras L, $L/\varphi(L)$ is φ-free. We shall denote by Φ the class of φ-free Lie algebras over the field F.

The purpose of this section is to answer the following question: Is the class Φ closed under ideal lattice isomorphisms? In order to obtain the answer, we shall investigate the structure of the Lie algebras over a field of characteristic zero which are not in Φ whose lattice of ideals is isomorphic to that of some Lie algebra in Φ. As a corollary we shall show that the answer to our question is negative unless F is an algebraically closed field of characteristic zero. As every finite dimensional Lie algebra can be mapped onto a φ-free one, we shall show that the results of this section provide information about the general problem of ideal lattice isomorphisms.

Before proving the main result in this section we shall need a series of lemmas, the first of which was obtained in [2].

3.1. Lemma [2]. *Let L be a Lie algebra over an arbitrary field F such that* $\dim L^2 = 1$ *and $L^2 \leq Z(L)$. Then, there exists a basis $\{a_1, \ldots, a_n, f_1, \ldots, f_n, z_1, \ldots, z_r\}$ for L such that $[a_i, f_i] = z_r$ and all other products are zero.*

Proof. Let V be a complementary subspace of L^2 in L, i.e., $L = L^2 \dotplus V$. Write $L^2 = F(z)$ and define $G: V \times V \to F$ by $g(v, w) := \lambda_{vw}$, where λ_{vw} is such that $[v, w] = \lambda_{vw} z$. As G is an alternate and not identically zero scalar product, by [5, p. 161], V has a basis

$\{a_1, \dots, a_n, f_1, \dots, f_n, z_1, \dots, z_{r-1}\}$ satisfying $G(a_i, f_i) = 1$ and $G(p, q) = 0$ otherwise. This proves the result. ∎

3.2. LEMMA. *Let L be a Lie algebra over an arbitrary field such that* $\mathrm{Rad}(L)^\infty$ *is abelian. If N is a minimal abelian ideal of L, then either* $N \le \mathrm{Rad}(L)^\infty$ *or* $N \le Z(\mathrm{Rad}(L))$.

Proof. We start with the following observation: From [15, Theorem 4.4.1.1], $\mathrm{Rad}(L)$ has Cartan subalgebras and they are exactly those subalgebras complementary to $\mathrm{Rad}(L)^\infty$. Then, for every Cartan subalgebra H of $\mathrm{Rad}(L)$ we have

$$\mathrm{Rad}(L) = \mathrm{Rad}(L)^\infty \dotplus H. \tag{3.2.1}$$

Note that the Fitting null and one component of $\mathrm{Rad}(L)$ relative to $\mathrm{ad}_L H$ are H and $\mathrm{Rad}(L)^\infty$ respectively. Assume N is not contained in $\mathrm{Rad}(L)^\infty$. The minimality of N implies $N \cap \mathrm{Rad}(L)^\infty = 0$. Now consider $N = N_0 \dotplus N_1$ the Fitting decomposition of N with respect to $\mathrm{ad}_L H$. It is immediate that $N_0 \le H$ and $N_1 \le \mathrm{Rad}(L)^\infty$. Since $N \cap \mathrm{Rad}(L)^\infty = 0$, we obtain $N \le H$ and therefore $N \cap Z(H) \neq 0$ because H is nilpotent. Take a nonzero element $x \in N \cap Z(H)$. From (3.2.1), $[x, \mathrm{Rad}(L)] = [x, \mathrm{Rad}(L)^\infty] \le N \cap \mathrm{Rad}(L)^\infty = 0$. Thus $N \cap Z(\mathrm{Rad}(L)) \neq 0$ and this yields $N \le Z(\mathrm{Rad}(L))$ by the minimality of N. ∎

3.3. LEMMA. *Let L be a Lie algebra over a field of characteristic zero and H be a Cartan subalgebra of* $\mathrm{Rad}(L)$. *For every $h \in H$, denote by $R_1(h)$ the Fitting one component of* $\mathrm{Rad}(L)$ *with respect to* $\mathrm{ad}\, h$. *Then, the following hold*:

(1) *If* $\mathrm{Rad}(L)^\infty$ *is abelian*, $\mathrm{Asoc}(L) \cap H = Z(\mathrm{Rad}(L))$. *Moreover, if N is a minimal ideal of L, then the minimal polynomial of* $\mathrm{ad}\, h|_N$ *is irreducible. In particular, either $N \le C_L(h)$ or $N \cap C_L(h) = 0$.*

(2) *If* $\mathrm{Rad}(L)^2 \le \mathrm{Asoc}(L)$, *then*:

(i) $\mathrm{Asoc}(L) = \mathrm{Rad}(L)^\infty \oplus Z(\mathrm{Rad}(L))$.

(ii) $\mathrm{Rad}(L)^2 \cap Z(\mathrm{Rad}(L)) = H^2$.

(iii) *If $h \notin \mathrm{Nil}(L)$, then $R_1(h)$ is a nonzero ideal equal to the sum of all minimal abelian ideals which are not contained in $C_L(h)$.*

Proof. (1) We claim that $Z(\mathrm{Rad}(L))$ is contained in $\mathrm{Asoc}(L)$. The result is immediate if L solvable; assume then that it is not the case and let S be a Levi factor of L. As $Z(\mathrm{Rad}(L))$ is an abelian ideal, we can decompose $Z(\mathrm{Rad}(L)) = Z_1 \oplus \cdots \oplus Z_n$, where each Z_i is an irreducible $\mathrm{ad}_L S$-module. It follows that each Z_i is a minimal abelian ideal, which proves our claim. Since $Z(\mathrm{Rad}(L)) \le H$ for every Cartan subalgebra H of $\mathrm{Rad}(L)$, from Lemma 3.2 and (3.2.1) we conclude that $\mathrm{Asoc}(L) \cap H =$

$Z(\text{Rad}(L))$. The second assertion is immediate for simple minimal ideals. Then, we shall restrict our attention to the case of minimal abelian ideals. Let $L = \text{Rad}(L) \dotplus S$ be a Levi decomposition of L and take $a \in L$ and $h \in H$. From (3.2.1), $a = a_1 + a_2 + s$, where $a_1 \in \text{Rad}(L)^\infty$, $a_2 \in H$, and $s \in S$. Now, if $x \in \text{Asoc}(L)$ it is immediate that $[x, a_1] = [[x,h], a_1] = [x,[h, a_2]] = [x,[h,s]] = 0$ (see [4, p. 91; 11, Proposition 8]). Thus, by the Jacobi identity we obtain

$$\begin{aligned}[][[x,h],a] &= [[x,h],a_2] + [[x,h],s] \\ &= [[x,a_2],h] + [[x,s],h] = [[x,a],h]. \quad (3.3.1)\end{aligned}$$

Next let $\rho\colon L \to gl(\text{Asoc}(L))$ be defined by $\rho(a) := \text{ad}\, a|_{\text{Asoc}(L)}$. Note that $\rho(L)$ is a completely reducible Lie algebra of linear transformations in $\text{Asoc}(L)$. Moreover, (3.3.1) implies $\rho(H) \leq Z(\rho(L))$. Then, from [4, p. 81], we conclude that the elements of $\rho(H)$ are semisimple. Now, let N be a minimal abelian ideal, $h \in H$, and $\mu(X)$ be the minimum polynomial of $\text{ad}\, h$ in N. Then, $\mu(X) = \pi_1(X)\ldots\pi_r(X)$, where $\pi_i(X)$ are distinct irreducible polynomials for $1 \leq i \leq r$. Consequently, we can decompose $N = N_{\pi_1} \dotplus \cdots \dotplus N_{\pi_r}$, where $N_{\pi_i} = \{x_i \in N : \pi_i(\text{ad}\, h)(x_i) = 0\}$ for $1 \leq i \leq r$. Since $\rho(H) \leq Z(\rho(L))$, from [4, p. 40] each N_{π_i} is a $\rho(L)$-invariant subspace. By the minimality of N we conclude that $r = 1$, which proves the second assertion in (1).

(2) Part (i) follows from Lemma 3.2, (3.2.1), and the first statement in (1). Now, from (3.2.1), $\text{Rad}(L)^2 = \text{Rad}(L)^\infty \dotplus H^2$; thus (ii) is a consequence of (1). To prove (iii), first we observe that $z \in \text{Nil}(L)$ if and only if $R_1(z) = 0$. Thus, $R_1(h) \neq 0$ because $h \notin \text{Nil}(L)$. Denote by S the sum of all minimal abelian ideals not contained in $C_L(h)$. Given P a minimal abelian ideal, from (1) we have $[P,h] = P$ if P is contained in S and $[P,h] = 0$ otherwise. Thus, $S \leq R_1(h) \leq \text{Rad}(L)^\infty$. As $\text{Rad}(L)^\infty$ is contained in $\text{Asoc}(L)$, we can take an ideal K such that $\text{Rad}(L)^\infty = S \oplus K$. From the above, $[K,h] = 0$. Since $\text{ad}\, h$ acts non-singularly on $R_1(h)$, we conclude that $R_1(h) = S$, which proves (iii). ∎

3.4. THEOREM. *Let L and M be Lie algebras over a field F of characteristic zero such that L is φ-free and M is not. Let H and $\tilde{H}$ be Cartan subalgebras of* Rad(M) *and* Rad(L), *respectively. If there exists an isomorphism α from $\Im(M)$ onto $\Im(L)$, then*:

(1) *α satisfies the following conditions*:

(i) $\alpha(M^2) = L^2$.

(ii) $\alpha(\text{Rad}(M)) = \text{Rad}(L)$.

(iii) $\alpha(\text{Asoc}(M)) = \text{Asoc}(L)$.

(iv) $\alpha(\varphi(M)) \leq \mathrm{Rad}(L)^{\infty}$.

(2) *M satisfies the following conditions*:

(i) $\mathrm{Rad}(M) = \mathrm{Rad}(M)^{\infty} \dot{+} H$.

(ii) $J(M) = \mathrm{Asoc}(M) = \mathrm{Rad}(M)^{\infty} \oplus Z(\mathrm{Rad}(M))$.

(iii) $\varphi(M) = Z(M) = H^2$.

(iv) $[h, H] = H^2$ *for every* $h \in H \setminus Z(\mathrm{Rad}(M))$. *In particular*, $Z(H) = Z(\mathrm{Rad}(M))$.

(v) $\dim M/M^2 = 2n$ *with* $n \geq 1$.

(vi) *Every solvable ideal which is not contained in* $\mathrm{Asoc}(M)$ *contains* $Z(M)$.

(vii) *If* S *and* $\tilde{S}$ *are semisimple Levi factors of* M *and* L, *then* S *and* $\tilde{S}$ *have the same number of simple ideals.*

(3) *L satisfies the following conditions*:

(*i) $\mathrm{Rad}(L) = \mathrm{Rad}(L)^{\infty} \dot{+} \tilde{H}$.

(*ii) $J(L) = \mathrm{Asoc}(L) = \mathrm{Nil}(L) = \mathrm{Rad}(L)^{\infty} \oplus Z(\mathrm{Rad}(L))$.

(iii) $Z(L) = 0$.

(*iv) $\tilde{H}$ *is abelian.*

(v) $\dim L/L^2 = \dim M/M^2$.

(vi) $\mathfrak{R} = \bigcap\{R_1(h) : h \in \tilde{H} \setminus Z(\mathrm{Rad}(L))\}$ *is a nonzero ideal of* L.

(vii) *If* N *is a minimal ideal contained in* $\mathfrak{R}$, *the set* $\mathfrak{S}_N = \{f \in \mathrm{ad}_N \tilde{H} : f \text{ is split}\}$ *is a subalgebra of* $\mathrm{ad}_N \tilde{H}$ *with dimensionality at most one. In particular*, F *is not algebraically closed.*

(viii) *If* L *is solvable, then* $\dim \tilde{H} \leq \dim N$ *for every minimal ideal* N *contained in* $\mathfrak{R}$.

($*$) *Parts* (3.i) *and* (3.iv) *and* $\mathrm{Asoc}(L) = \mathrm{Nil}(L) = \mathrm{Rad}(L)^{\infty} \oplus Z(\mathrm{Rad}(L))$ *hold for any* φ*-free Lie algebra over a field of characteristic zero.*

Proof. First, we shall prove the final remark ($*$). If L_1 is a φ-free Lie algebra, by Theorem 7.4 in [12], $\mathrm{Nil}(L_1) = \mathrm{Asoc}(L_1)$. Then, $\mathrm{Rad}(L_1)^2 \leq \mathrm{Asoc}(L_1)$ because $\mathrm{Rad}(L_1)^2$ is nilpotent, and therefore $\mathrm{Rad}(L_1)^{\infty}$ is abelian. Thus (3.i) follows from [15, Theorem 4.4.1.1], and Lemma 3.3 implies $\mathrm{Asoc}(L_1) = \mathrm{Rad}(L_1)^{\infty} \oplus Z(\mathrm{Rad}(L_1))$. In addition, if T is a Cartan subalgebra of $\mathrm{Rad}(L_1)$, (2.ii) of Lemma 3.3 and [7, Sect. 4] yield $T^2 \leq \varphi(\mathrm{Rad}(L_1))$. Now, Corollary 4.2 in [12] implies $T^2 \leq \varphi(L_1) = 0$, proving (3.iv). Hence, ($*$) holds.

Now we shall prove the rest of the statements. As L is φ-free and $J(L)$ is nilpotent, from the first paragraph we conclude that $J(L) \leq \mathrm{Soc}(L)$.

Then, $\alpha^{-1}(J(L)) = J(M) \leq \mathrm{Soc}(M)$ and the nilpotency of $J(M)$ implies $J(M) \leq \mathrm{Asoc}(M)$.

(1.i), (1.ii), (1.iii), (2.vii), (3.v) Since $J(M) = M^2 \cap \mathrm{Rad}(M)$, from (i) of Theorem 2.4, $\Im(M/J(M))$ is complemented. We claim that the interval $[J(M): M]$ is not distributive. If it is not the case, applying (iii) of Theorem 2.4, we obtain that either $J(M) = \mathrm{Rad}(M)$ or $J(M)$ has codimension one in $\mathrm{Rad}(M)$. As $J(M) \leq \mathrm{Asoc}(M)$, it follows that $\mathrm{Asoc}(M) = \mathrm{Rad}(M)$ or $\mathrm{Rad}(M) = \mathrm{Asoc}(M) \dotplus F(x)$. In the first case and in the second one if M is solvable, we conclude that M splits over its zerosocle, a contradiction (see Theorem 7.3 in [12]). Therefore M is not solvable and $\dim \mathrm{Rad}(M)/\mathrm{Asoc}(M) = 1$. Let S be a Levi factor of M. Then, there exists a one-dimensional $\mathrm{ad}_M S$-module B such that $\mathrm{Rad}(M) = \mathrm{Asoc}(M) \dotplus B$. As $[B, S] = 0$, we get M splits over its zerosocle, a contradiction. Hence the interval $[J(M): M]$ is not distributive and therefore (iii) of Theorem 2.4 implies that the center of $M/J(M)$ has dimensionality at least two. Consequently, $\dim M/M^2 > 1$ and now the results follow from Corollary 2.9.

(2.i), (2.ii), (2.iii) As $\mathrm{Rad}(M)^2 \leq J(M) \leq \mathrm{Asoc}(M)$, (2) of Lemma 3.3 implies $\mathrm{Asoc}(M) = \mathrm{Rad}(M)^\infty \oplus Z(\mathrm{Rad}(M))$. Then, $\mathrm{Rad}(M)^\infty$ is abelian and therefore [15, Theorem 4.4.1.1] yields $\mathrm{Rad}(M) = \mathrm{Rad}(M)^\infty \dotplus H$. To prove (2.ii) and (2.iii), we shall consider two cases:

First Case. M is solvable. Then, $J(M) = M^2 = M^\infty \dotplus H^2$ and $\mathrm{Asoc}(M) = M^\infty \oplus Z(M)$. If $H^2 = 0$, we can take a complementary subalgebra C to $Z(M)$ in H which is a complementary subalgebra to $\mathrm{Asoc}(M)$ in M. It follows that M splits over its zerosocle, a contradiction (see Theorem 7.3 in [12]). Hence $H^2 \neq 0$ and Lemma 3.3 implies $H^2 \leq Z(M)$. We claim that $H^2 = Z(M)$. Suppose it is not the case and take nonzero elements $a \in H^2$ and $b \in Z(M) \setminus H^2$. Then, $F(a)$, $F(b)$, and $F(a+b)$ are minimal ideals of M satisfying

$$F(a) \oplus F(b) = F(a) \oplus F(a+b) = F(b) \oplus F(a+b).$$

From (1.i), $\alpha(F(a))$ is contained in L^2 and $\alpha(F(b))$ is not. If $\alpha(F(a+b))$ is contained in L^2, we get $\alpha(F(b)) \leq \alpha(F(a)) \oplus \alpha(F(a+b)) \leq L^2$, a contradiction. Note that from (1.ii) and (1.iii), L is solvable and L^2 is contained in $\mathrm{Asoc}(L)$, so that L^∞ is abelian. Now, from Lemma 3.2 we conclude that $\alpha(F(b))$ and $\alpha(F(a+b))$ are central minimal ideals. Thus, $\alpha(F(a)) \leq Z(L) \cap L^2 \leq \varphi(L) = 0$ (see [7, Sect. 4]), which is a contradiction. This proves our claim and therefore (2.ii) holds. On the other hand, we observe that $\mathrm{Asoc}(M/Z(M))$ has a complement in $M/Z(M)$. Then, Theorem 7.3 and Corollary 4.4 in [12] imply that $\varphi(M) \leq Z(M)$. Since

$Z(M) \leq M^2$, from [7, Sect. 4], $Z(M) \leq \varphi(M)$, which completes the proof of (2.iii).

Second Case. M is not solvable. Let S be a Levi factor of M, i.e., $M = \mathrm{Rad}(M) \dotplus S$. As $\mathrm{Rad}(M)$ is an $\mathrm{ad}_M S$-completely reducible module, there exists an $\mathrm{ad}_M S$-module A such that $\mathrm{Rad}(M) = J(M) \dotplus A$. Note that $A \neq 0$ because $J(M) \neq \mathrm{Rad}(M)$. Since $[A, S] \leq A \cap J(M) = 0$ (see [7, Theorem 1]), we conclude that A is contained in $C_M(S)$ and we can write $\mathrm{Rad}(M) = \mathrm{Asoc}(M) + C_M(S)$. Now consider $C = \mathrm{Asoc}(M) \cap C_M(S)$. As $[C, M] = [C, \mathrm{Rad}(M)] = [C, C_M(S)] \leq C$, we have that C is an ideal of M contained in $\mathrm{Asoc}(M)$. Then, we can decompose $\mathrm{Rad}(M) = K \dotplus C_M(S)$, where K is an ideal such that $\mathrm{Asoc}(M) = K \oplus C$. Denote $K \dotplus S$ by P and observe that P is an ideal of M ($[P, \mathrm{Rad}(M)] \leq K$ and $[P, S] \leq P$) and $\varphi(M)$ is not contained in P. Since $\varphi(M/P) = (\varphi(M) + P)/P$ (see [14, Theorem 3]), it follows that $C_M(S)$ is a solvable and not φ-free Lie algebra whose lattice of ideals is isomorphic to $\mathfrak{I}(L/\alpha(P))$ and by [14, Theorem 3], $L/\alpha(P)$ is φ-free. Thus from the first case, $\mathrm{Asoc}(C_M(S)) = C_M(S)^\infty \oplus Z(C_M(S))$ and $0 \neq E^2 = Z(C_M(S))$, where E is a Cartan subalgebra of $C_M(S)$. Note that $E^2 = Z(M) \leq \mathrm{Rad}(M)^2 \leq J(M)$. Moreover, we can decompose M as follows: $M = K \dotplus C_M(S)^\infty \dotplus E \dotplus S$. It is easily checked that $\mathrm{Asoc}(M/Z(M))$ has a complement in $M/Z(M)$; therefore we conclude that $\varphi(M) \leq Z(M)$ (see Theorem 7.3 and Corollary 4.4 in [12]). Finally, if N is a minimal abelian ideal not contained in $J(M)$, $[N, M] \leq N \cap J(M) = 0$ and then $N \leq Z(M) \leq J(M)$, a contradiction. As $J(M) \leq \mathrm{Asoc}(M)$, it follows that $J(M) = \mathrm{Asoc}(M)$, proving (2.ii).

Now we shall prove (2.iii). The above paragraph shows that $\varphi(M) \leq Z(M) \leq \mathrm{Rad}(M)^2$. Then, [7, Sect. 4] implies $Z(M) \leq \varphi(\mathrm{Rad}(M))$ and, from Corollary 4.2 in [12], we conclude that $Z(M) = \varphi(M)$. Moreover, (2.ii) of Lemma 3.3 implies $Z(M) \leq H^2$. Next, take $s \in S$, $h_1, h_2 \in H$. By the Jacobi identity we have

$$[[h_1, h_2], s] = [[h_1, s], h_2] + [h_1, [h_2, s]].$$

From Theorem 1 in [7] and (2.ii), we obtain $[h_i, s] \in \mathrm{Rad}(M)^\infty \oplus Z(\mathrm{Rad}(M))$ and therefore $[[h_1, s]h_2], [h_1[h_2, s]] \in \mathrm{Rad}(M)^\infty$. As $H^2 \leq Z(\mathrm{Rad}(M))$ (see Lemma 3.3), we get $[[h_1, h_2], s] \in \mathrm{Rad}(M)^\infty \cap Z(\mathrm{Rad}(M)) = 0$. Hence $H^2 \leq Z(M)$, which completes the proof of (2.iii).

(2.iv) Let $h \in H \setminus Z(\mathrm{Rad}(M))$ and set $P = [h, H] = \{[h, x] : x \in H\}$. As P is contained in H^2, from (2.iii), P is a central ideal of M. Then, we can take ideals N and Q of M such that $Z(\mathrm{Rad}(M)) = P \oplus N \oplus Q$ and $P \oplus N = H^2$. Now consider the Lie algebra M/K, where $K = \mathrm{Rad}(M)^\infty \oplus P \oplus Q$. Note that $\mathrm{Rad}(M/K) = \mathrm{Rad}(M)/K$ is nilpotent and therefore $\varphi(M/K) = \text{Rad}(M/K)^2$ from Corollary 7.8 in [12]. Moreover, $J(M/K) = J(M)/K$ because $K \leq J(M)$. If $N \neq 0$, M/K is a not φ-free Lie algebra whose lattice of ideals is isomorphic to $\Im(L/\alpha(K))$. Thus, (2.ii) implies $J(M)/K = Z(\text{Rad}(M/K))$. As $h + K \in Z(\text{Rad}(M/K))$, we conclude that $h \in J(M) \cap H$. Now from (1) of Lemma 3.3, we get $h \in Z(\text{Rad}(M))$, a contradiction. Hence $N = 0$ and therefore (2.iv) holds. The last assertion is easily checked.

(2.v) Let $x \in H^2$ and Q be a complement vector space of $F(x)$ in $Z(\text{Rad}(M))$. Consider the Lie algebra H/Q and note that $(H/Q)^2 = Z(\text{Rad}(M))/Q$ is a one-dimensional Lie algebra contained in $Z(H/Q)$. Now, let $h + Q$ be a nonzero element in $Z(H/Q)$. From (2.iv) it is easy to deduce that $h \in Z(\text{Rad}(M))$ and therefore $Z(H/Q) = (H/Q)^2$. Now, applying Lemma 3.1 we see that $\dim(H/Q)/(H/Q)^2 = \dim H/Z(\text{Rad}(M))$ is even. On the other hand, we have $M/M^2 \cong \text{Rad}(M)/J(M) = (J(M) + H)/J(M)$. From (2.ii) of this theorem and (1) of Lemma 3.3, $J(M) \cap H = Z(\text{Rad}(M))$. Thus $M/M^2 \cong H/Z(\text{Rad}(M))$, which proves (2.v).

(2.vi) Let N be a solvable ideal which is not contained in $\text{Asoc}(M)$. Consider $N = N_0 \dotplus N_1$ the Fitting decomposition of N with respect to $\text{ad}_M H$. Note that $N_0 \leq H$ and $N_1 \leq \text{Rad}(M)^\infty$ (see (3.2.1)). As N is not contained in $\text{Asoc}(L)$, there exists an element $h \in N_0 \setminus Z(\text{Rad}(M))$. Then, from (2.iv) and (2.iii), we get $N \geq [h, H] = H^2 = Z(M)$.

(3.i), (3.ii), (3.iii), (3.iv) From the remark $(*)$ and the statements (1.iii) and (2.ii), it remains only to prove (3.iii). We observe that $Z(L) \leq J(L) \leq L^2$. Then, [7, Sect. 4] implies $Z(L) = 0$.

(1.iv), (3.vi) From Theorem 7.5 in [12], we have the following decomposition: $L = \text{Asoc}(L) \dotplus (E \oplus S)$, where E is an abelian subalgebra and S is a semisimple one. Then, $H_1 = Z(\text{Rad}(L)) \dotplus E$ is a Cartan subalgebra of $\text{Rad}(L)$ (use (3.ii) and [15, Theorem 4.4.1.1]). Note that $J(L) \neq \text{Rad}(L)$ because $J(M) \neq \text{Rad}(M)$ and therefore E is a nonzero subalgebra. Take $0 \neq z \in E$ and observe that $z \notin \text{Nil}(L)$. From (2.iii) of Lemma 3.3 it is immediate that $T = R_1(z) \dotplus ((z))$ is a solvable ideal of L not contained in $\text{Asoc}(L)$. Thus (1.ii), (1.iii), (2.iii), and (2.vi) imply $0 \neq \alpha(\varphi(M)) \leq T \cap \text{Asoc}(L) = R_1(z) \leq \text{Rad}(L)^\infty$, proving (1.iv). Now take $h \in H_1 \setminus Z(\text{Rad}(L))$. Then, $h = b + z$, where $b \in Z(\text{Rad}(L))$ and z is a nonzero element of E. Clearly, $R_1(h) = R_1(z)$ and therefore $\bigcap\{R_1(h) : h \in H_1 \setminus Z(\text{Rad}(L))\} = \bigcap\{R_1(z) : z \in E \setminus \{0\}\} \neq 0$. On the other hand, from [15, Theorem 4.4.1.1], $\tilde{H} = (1 + \text{ad}\, x)(H_1)$ for some $x \in \text{Rad}(L)^\infty$. Let $h \in \tilde{H} \setminus Z(\text{Rad}(L))$ and $z \in H_1 \setminus Z(\text{Rad}(L))$ be such that $h = z + [z, x]$. As $[[z, x], a] = 0$ for every $a \in \text{Rad}(L)^\infty$, we conclude that $C_L(h) \cap \text{Asoc}(L) = C_L(z) \cap \text{Asoc}(L)$ and therefore $R_1(h) = R_1(z)$ by (2.iii) of Lemma 3.3. This implies the statement (3.vi).

(3.vii) Let N be a minimal ideal contained in $\mathfrak{R}$. Since $\tilde{H}$ is abelian, $\mathrm{ad}_N \tilde{H}$ is a set of commuting linear transformations on N. From Lemma 3.3, for each $h \in \tilde{H} \setminus Z(\mathrm{Rad}(L))$, $\mathrm{ad}\, h$ acts nonsingularly on N with irreducible minimal polynomial, so that $\mathrm{ad}_N \tilde{H}$ is an abelian Lie algebra of linear transformations isomorphic to $\tilde{H}/Z(\mathrm{Rad}(L))$. We remark that $L/L^2 \cong \tilde{H}/Z(\mathrm{Rad}(L))$. If $g \in \mathfrak{S}_N$ it is clear that $g|_N = \alpha I$ for some $\alpha \in F \setminus \{0\}$ and therefore $\mathfrak{S}_N$ is a subalgebra of $\mathrm{ad}_N \tilde{H}$. Next, assume $\dim \mathfrak{S}_N \geq 2$ and take $f, g \in \mathfrak{S}_N$ linearly independent transformations. Then, $f = \mathrm{ad}\, h_1|_N$ and $g = \mathrm{ad}\, h_2|_N$, where $h_i \in \tilde{H} \setminus Z(\mathrm{Rad}(L))$ and $f = \alpha I$, $g = \beta I$ for some $\alpha, \beta \in F \setminus \{0\}$. As $\beta f - \alpha g$ acts trivially on N, it follows that N is not contained in $R_1(\beta h_1 - \alpha h_2)$ and therefore $\beta h_1 - \alpha h_2 \in Z(\mathrm{Rad}(L))$. We conclude that $\beta f = \alpha g$, a contradiction. Hence $\dim \mathfrak{S}_N \leq 1$. If F is algebraically closed, $\mathfrak{S}_N = \mathrm{ad}_N \tilde{H}$. Since $\dim(\mathrm{ad}_N \tilde{H}) = \dim L/L^2 > 1$, the last assertion is clear.

(3.viii) If L is solvable, $Z(\mathrm{Rad}(L)) = Z(L) = 0$ by (3.iii). Then, for each N minimal ideal contained in $\mathfrak{R}$ we have $\tilde{H} \cong \mathrm{ad}_N \tilde{H} \leq \mathrm{End}_F(N)$. On the other hand, by the minimality of N, proper $\mathrm{ad}_N \tilde{H}$-invariant subspaces do not exist. As $\mathrm{ad}_N \tilde{H}$ is a subspace of commuting linear transformations on N, from Proposition 2 in [10], we have that $\mathrm{ad}_N \tilde{H}$ is contained in a maximal subfield of $\mathrm{End}_F(N)$ and this implies $\dim(\mathrm{ad}_N \tilde{H}) = \dim \tilde{H} \leq \dim N$. ▮

We observe that if L and M are Lie algebras as in Theorem 3.4, L has at least one minimal abelian ideal contained in $\mathrm{Rad}(L)^{\infty}$. Moreover, it is easy to verify that L and M can be mapped onto Lie algebras L_1 and M_1, respectively, with isomorphic lattice of ideals such that L_1 is φ-free, having a unique minimal abelian ideal, and M is not φ-free. The following corollary describes the complete structure of the family of couples (L_1, M_1).

3.5. COROLLARY. *Let L and M be Lie algebras over a field F of characteristic zero. Suppose L is φ-free and has a unique minimal abelian ideal and M is not φ-free. Then, the lattices $\mathfrak{I}(L)$ and $\mathfrak{I}(M)$ are isomorphic if and only if one of the following holds*:

(1) (i) *M is the Lie algebra with basis $\{a_1, \ldots, a_n, f_1, \ldots, f_n, z\}$ and products $[a_i, f_j] = \delta_{ij} z$, $[a_i, z] = [f_i, z] = 0$.*

(ii) *$L = V \dotplus T$, where V is a nonzero abelian ideal, T is a $2n$-dimensional abelian subalgebra, $\mathrm{ad}\, x|_V$ is non-singular for every $x \in T \setminus \{0\}$, and V is an $\mathrm{ad}_L T$-irreducible module.*

(2) (i) *$M = A \oplus S$, where A is an algebra of the type described in* (1.i) *and S is a semisimple ideal.*

(ii) *$L = B \oplus K$, where B is an algebra of the type described in* (1.ii) *and K is a semisimple ideal with the same number of simple ideals as S.*

If F is algebraically closed, there are no Lie algebras of the type described in (1.ii).

Proof. From Theorem 3.4 and Lemma 3.1, we see that Rad(M) has a basis as in (1.i). Suppose first that M is solvable. Then, Theorem 3.4 implies L is solvable, dim $L/L^2 = 2n$, and Asoc(L) $= L^2$. Moreover $L^2 = L^\infty$ and therefore $L = L^2 \dotplus H$, where H is a Cartan subalgebra of L. Now, by Lemma 3.3 we find that every nonzero element of H acts non-singularly on L^2. Finally, the minimality of L^2 yields L^2 is an $\mathrm{ad}_L\, H$-irreducible module. Hence M and L are as in (1).

Next, assume M is not solvable and let S be a Levi factor of M. From Theorem 3.4, $J(M) = Z(M)$ and then, it is easily checked that S is a semisimple ideal of M. By using (1.ii) of Theorem 3.4, it is clear that $L = \mathrm{Rad}(L) \oplus K$, where K is a semisimple ideal. Thus, from the above paragraph and (2.vii) of Theorem 3.4, M and L are as in (2).

To prove the converse, assume first that M and L are as in (1). Since M is nilpotent and $M^2 = Z(M) = F(z)$, every nonzero ideal of M contains M^2. On the other hand, we observe that $V = L^2$ and it is the only minimal ideal of L. Thus every nonzero ideal of L contains L^2. Moreover, the intervals $[L^2 : L]$ of $\Im(L)$ and $[M^2 : M]$ of $\Im(M)$ are isomorphic. Hence we conclude that $\Im(L) \cong \Im(M)$. Finally, suppose L and M are as in (2) and observe that from the above there exists a lattice isomorphism α from $\Im(\mathrm{Rad}(L))$ onto $\Im(\mathrm{Rad}(M))$. As K and S are semisimple with the same number of simple ideals, by Corollary 2.8 there exists a lattice isomorphism β from $\Im(K)$ onto $\Im(S)$. Note that every ideal P of L (resp. Q of M) has a unique Levi decomposition $P = P \cap \mathrm{Rad}(L) \oplus P \cap K$ (resp. $Q = Q \cap \mathrm{Rad}(M) \oplus Q \cap S$). Therefore, $\alpha \oplus \beta$: $\Im(L) \to \Im(M)$ defined by $\alpha \oplus \beta(P) = \alpha(P \cap \mathrm{Rad}(L)) \oplus \beta(P \cap K)$ is a lattice isomorphism. ∎

In the case $F = \mathbb{R}$, the above result becomes:

3.6. Corollary. *Let L and M be real Lie algebras. Suppose L is φ-free and has a unique minimal abelian ideal and M is not φ-free. Then, the lattices $\Im(L)$ and $\Im(M)$ are isomorphic if and only if one of the following holds*:

(1) (i) *M is the Lie algebra with basis $\{a, b, z\}$ and products $[a, b] = z$.*

(ii) *L is the Lie algebra with basis $\{a, b, x, y\}$ and products $[a, x] = a$, $[b, x] = b$, $[a, y] = b$, $[b, y] = -a$.*

(2) (i) *$M = A \oplus S$, where A is an algebra as in* (1.i) *and S is a semisimple ideal.*

(ii) *$L = B \oplus K$, where B is an algebra as in* (1.ii) *and K is a semisimple ideal with the same number of simple ideals as S.*

In particular, given L and M real Lie algebras having isomorphic lattices of ideals such that L is φ-free and M is not, if H is a Cartan subalgebra of Rad(M), *H has a basis* $\{a, b, z_1, \dots, z_r\}$ *such that* $[a, b] = z_r$ *and all other products are zero. Moreover, if M is solvable* $r = 1$.

Proof. We have that L and M are as in (1) or (2) of Corollary 3.5. We can restrict our attention to the case L and M as in (1). In that case, T is a Cartan subalgebra of L and V is the only one minimal ideal. As every nonzero element of $\mathrm{ad}_L T$ acts non-singularly on V, from (2.viii) of Theorem 3.4, $\dim V \geq \dim T \geq 2$, and by [10, Proposition 2], $\mathrm{ad}_V T$ is contained in a field extension of the real numbers inside $\mathrm{End}_{\mathbb{R}}(V)$. Thus $T = \mathbb{R}\langle x, z\rangle$ and ad x acts as the identity on V. Since V is a minimal ideal of dimensionality at least 2, from Lemma 3.3 we conclude that the minimal polynomial of ad $z|_V$, denoted by $\pi(X)$, is an irreducible quadratic. Then, $\dim V = 2m$. If $m > 1$, it is easily checked that V has a proper subspace invariant under $\mathrm{ad}_V T$, a contradiction. Hence, $\dim V = 2$. Now, write $\pi(X) = X^2 - \beta X - \alpha$, where $\alpha, \beta \in \mathbb{R}$, and let $\{a_1, a_2\}$ be a basis for V such that $[a_1, z] = a_2$ and $[a_2, z] = \alpha a_1 + \beta a_2$. Next, consider $y = t_0 x + t_1 z$, where $t_1 = \sqrt{-4/(\beta^2 + 4\alpha)}$ and $t_0 = -\beta t_1/2$. The minimal polynomial of ad $y|_V$ is $X^2 + 1$. Thus picking a basis $\{a, b\}$ for V corresponding to the canonical matrix of ad $y|_V$ we get that $\{a, b, x, y\}$ is a basis of L with products as in (1.ii). As $\dim T = 2$, from Corollary 3.5 we conclude that M is as in (1.i). The final assertion is straightforward. ∎

3.7. COROLLARY. *Let L and M be Lie algebras over a field F of characteristic zero. Suppose either F is algebraically closed or* $\dim L/L^2$ *is odd. Let* α *be a lattice isomorphism from* $\mathfrak{I}(L)$ *onto* $\mathfrak{I}(M)$. *Then*:

(i) $\alpha(\varphi(L)) = \varphi(M)$.

(ii) $L \in \Phi$ *if and only if* $M \in \Phi$.

(iii) *If* $\dim L/L^2 > 1$, $\alpha(\mathrm{Nil}(L)) = \mathrm{Nil}(M)$.

(iv) *If the dimensionality of L and M is at least* 2, *then L is nilpotent if and only if M is.*

Proof. (i), (ii) The second statement is an immediate consequence of the first one and therefore we need only prove (i). As $\varphi(L) \leq L^2$ and $L/\varphi(L)$ is φ-free, from Theorem 3.4 we see that $M/\alpha(\varphi(L))$ is φ-free and therefore $\alpha(\varphi(L)) \leq \varphi(M)$. A similar argument with the roles of L and M reversed and using α^{-1} shows that $\alpha(\varphi(L)) = \varphi(M)$.

(iii) Note that $\mathrm{Nil}(L)/\varphi(L) = \mathrm{Asoc}(L/\varphi(L)) = \mathrm{Nil}(L/\varphi(L))$ (see Proposition 4 in [11] and Theorem 7.4 in [12]) and the same is true changing L by M. Thus from (i), we can assume without loss of generality that $\varphi(L) = \varphi(M) = 0$. As $\dim L/L^2 > 1$, from Corollary 2.9 it follows that $\alpha(\text{Asoc}(L)) = \text{Asoc}(M)$, which proves (iii).

(iv) Note that if N is a nilpotent Lie algebra of dimensionality at least two, $\dim N/N^2 > 1$. Thus the result follows from (iii). ∎

Remark. The statement (iv) in Corollary 3.7 was previously obtained in [2], where a more detailed study of the lattices of ideals of nilpotent Lie algebras was carried out.

4. Is the Class Φ Closed under Lattice Isomorphisms?

It is clear from Section 3 that the class Φ is closed under ideal lattice isomorphisms if the base field is algebraically closed of characteristic zero. In this section we shall give examples showing that this is not true either for non-algebraically closed fields of any characteristic or for algebraically closed fields of positive characteristic. Some of these examples show that most of the statements established in Theorem 3.4 do not hold for positive characteristic, where the behaviour of φ-free Lie algebras is in general quite different from that in the case of characteristic zero.

4.1. Example. Let F be a field of arbitrary characteristic, V be an n-dimensional vector space, and $f(X) = X^n - \alpha_{n-1}X^{n-1} - \cdots - \alpha_1 X - \alpha_0$ be an irreducible polynomial over F with $\alpha_0 \neq 0$. Given basis $\{a_1, \ldots, a_n\}$ for V, we define the linear transformation $\rho\colon V \to V$ by $\rho(a_i) = a_{i+1}$ for $0 \le i \le n-1$ and $\rho(a_n) = \alpha_0 a_1 + \cdots + \alpha_{n-1}a_n$. Let T be the linear span of $\rho, \rho^2, \ldots, \rho^m$ with $m \le n$. We consider T as an m-dimensional abelian subalgebra of $gl(V)$ and form the Lie algebra $L = V \dot{+} T$ by defining the Lie bracket in L in the following way: $[x, y] = 0$ for $x, y \in V$, $[x, \rho^i] = \rho^i(x)$, and T keeps its original Lie bracket. It is immediate that V has no proper $\text{ad}_L T$-invariant subspaces and every nonzero element in T acts non-singularly on V. Then, it is easily checked that every nonzero ideal of L contains V and the interval $[V : L]$ is abelian of length m. In addition, L is φ-free because $V = \text{Asoc}(L)$. Next consider the nilpotent Lie algebra M with basis $\{a_1, \ldots, a_k, f_1, \ldots, f_k, z\}$ having as its only nonzero products $[a_i, f_i] = z = -[f_i, a_i]$. Then, $\varphi(M) = F(z)$ and therefore M is not φ-free. If $m = 2k$, it is clear that $\mathfrak{I}(L)$ and $\mathfrak{I}(M)$ are isomorphic. This example shows that the answer to the question formulated in this section is negative for Lie algebras over a non-algebraically closed field of arbitrary characteristic.

4.2. Example. Let F be a field of characteristic $p > 3$ and M be the $(p+2)$-dimensional Lie algebra over F with basis $\{a_1, \ldots, a_{p-1}, x, y, z\}$ and products $[x, y] = z$, $[x, z] = 2x + a_1$, $[y, z] = -2y$, $[a_i, z] = 2ia_i$

for $1 \leq i \leq p-1$, $[a_i, x] = ia_{i+1}$ for $1 \leq i \leq p-2$, $[a_{p-1}, x] = [a_1, y] = 0$, $[a_i, y] = ia_{i-1}$ for $2 \leq i \leq p-1$, and $[a_i, a_j] = 0$. It is easily checked that $\mathrm{Rad}(M) = F(a_1, \ldots, a_{p-1}) = \varphi(M)$ and this is the only minimal ideal of M. Then, the lattice of ideals of M is the three-element chain $0 < F(a_1, \ldots, a_{p-1}) < M$. Now consider the Lie algebra L with basis $\{x, y\}$ and product $[x, y] = x$. Clearly, L is φ-free and the lattice of ideals of L is the three-element chain $0 < F(x) < L$. Hence $\Im(L)$ and $\Im(M)$ are isomorphic. This example shows that the class Φ is not closed under ideal lattice isomorphisms if the base field is algebraically closed of positive characteristic. In addition, the statements (1.i), (1.ii), (2.iii), (2.v) in Theorem 3.4 fail in positive characteristic. Moreover, the Lie algebra $gl(V)/FI$ in Example 2.7 is φ-free and its lattice of ideals is isomorphic to $\Im(M)$; thus (1.iii) in Theorem 3.4 does not hold.

4.3. EXAMPLE. Let F be a field of characteristic $p > 0$ and L be the $(p+2)$-dimensional Lie algebra over F with basis $\{a_1, \ldots, a_p, x, y\}$ and products $[a_i, a_j] = 0$, $[a_i, x] = a_{i+1}$ for $1 \leq i \leq p-1$, $[a_p, x] = a_1$, $[a_i, y] = (i-1)a_i$ for $1 \leq i \leq p$, $[x, y] = x$. Note that L is a solvable Lie algebra and $L^2 = ((a_1, \ldots, a_p, x)) = L^\infty$ is the only maximal ideal of L. If we denote $((a_1, \ldots, a_p))$ by V, then $\mathrm{Nil}(L) = V$. As $\operatorname{ad} y|_V$ has minimal polynomial $X(X-1)\ldots(X-(p-1))$, $F(a_i)$ for $1 \leq i \leq p$ are the only $(\operatorname{ad} y)$-irreducible subspaces of V. From that it is easy to check that V is the only minimal ideal of L and therefore $V = \mathrm{Asoc}(L)$. Hence L is φ-free and the lattice of ideals of L is the four-element chain $0 < V < L^2 < L$.

Next consider the Lie algebra $\mathrm{gl}(V)$ as in Example 2.7 and let N be a maximal subalgebra of $gl(V)$. If FI is not contained in N, $gl(V) = N + FI$ and therefore $gl(V)^2 \leq N$, a contradiction. It follows that $FI \leq \varphi(gl(V))$, and as the Frattini ideal is nilpotent (see [12, Theorem 6.5]), we conclude that $FI = \varphi(gl(V))$. Hence $gl(V)$ is neither φ-free nor solvable and the lattice of ideals of $gl(V)$ is the four-element chain $0 < FI < gl(V)^2 < gl(V)$. Thus $\Im(L)$ and $\Im(gl(V))$ are isomorphic and we observe that (2.ii) and (3.ii) in Theorem 3.4 fail in positive characteristic.

4.4. EXAMPLE. Let F be a field of characteristic $p > 0$ and L be the $(p+3)$-dimensional Lie algebra over F with basis $\{a_1, \ldots, a_p, x, y, z\}$ and products $[x, a_1] = 0$, $[x, a_i] = a_{i-1}$ for $2 \leq i \leq p$, $[y, a_p] = 0$, $[y, a_i] = ia_{i+1}$ for $1 \leq i \leq p-1$, $[x, y] = z$, $[z, a_i] = a_i$ for $1 \leq i \leq p$, $[x, z] = [y, z] = [a_i, a_j] = 0$. We have that $L^2 = F(a_1, \ldots, a_p, z)$ and $V = F(a_1, \ldots, a_p) = L^\infty$ are the only maximal and minimal ideals of L, respectively, and the interval $[L^2 : L]$ is abelian of length two. We observe that L is φ-free, and $H = F(x, y, z)$ is a Cartan subalgebra of L which is not abelian (from (3.iv) of Theorem 3.4 this is not possible in characteristic

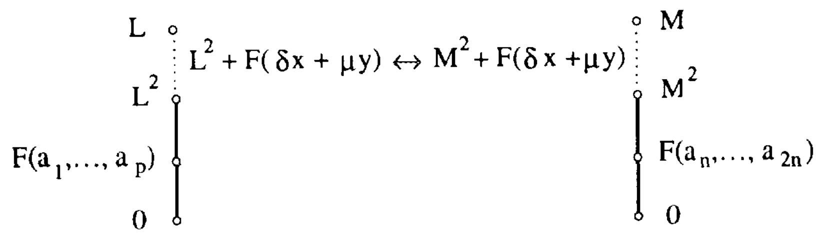

FIGURE 4.4.1

zero). In addition, L/V is a nilpotent and non-abelian Lie algebra, and thus L/V is not φ-free. This fact cannot occur in characteristic zero (see [14, Theorem 3]).

Next consider M the $2(n+1)$-dimensional Lie algebra with basis $\{a_1, \ldots, a_{2n}, x, y\}$ and products $[a_i, a_j] = [x, y] = 0$, $[a_i, x] = a_i$ for $1 \leq i \leq 2n$, $[a_i, y] = a_{i+1}$ for $i \notin \{n, 2n\}$, $[a_n, y] = \alpha_0 a_1 + \cdots + \alpha_{n-1} a_n + a_{n+1}$, and $[a_{2n}, y] = \alpha_0 a_{n+1} + \cdots + \alpha_{n-1} a_{2n}$, where $X^n - \alpha_{n-1} X^{n-1} - \cdots - \alpha_1 X - \alpha_0$ is an irreducible polynomial over F, $\alpha_0 \neq 0$, and $n > 1$. We have that $M^2 = F(a_1, \ldots, a_{2n})$ and $V = F(a_n, \ldots, a_{2n}) = \varphi(M)$ are the only maximal and minimal ideals of M, respectively, and the interval $[M^2 : M]$ is abelian of length 2. In Fig. 4.4.1 we show the lattices of ideals of L and M, which are clearly isomorphic. Hence, (3.vi), (3.vii), (3.viii) in Theorem 3.4 do not hold in positive characteristic.

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