Lie Algebras With a Small Number of Ideals

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ABSTRACT

We determine the Lie algebras such that their number of ideals is at most five. A complete classification is given of the solvable Lie algebras in this class over algebraically closed fields of characteristic zero and the real field.

1. INTRODUCTION

Let L be a Lie algebra. The ideals of L may be taken as the elements of a lattice $\Im(L)$ under the operations of sum and intersection. Although L determines $\Im(L)$ uniquely, in general $\Im(L)$ does not determine L uniquely. Moreover, there exist lattices $\mathscr{L}$ that are not the $\Im(L)$ for any Lie algebra L. For example, the 5-element lattice $\mathscr{L}_1$ represented by the diagram in Figure 1 cannot occur as the lattice of ideals of any Lie algebra, because $\mathscr{L}_1$ is nonmodular (see [2, p. 20]).

In the present paper we are interested in answering the following questions: (1) Which n-element lattices $\mathscr{L}$ with $1 \leqslant n \leqslant 5$ can occur as the lattice of ideals of some Lie algebra? (2) If $\mathscr{L}$ is such a lattice, how many Lie algebras L exist such that $\mathscr{L}$ is their lattice of ideals? In Section 2, we shall obtain the answer to the first question, which helps us to determine the

*Work supported in part by the I.E.R. (Instituto de Estudios Riojanos).

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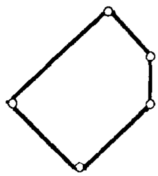

FIG. 1.

structure of the Lie algebras that have at most five ideals. These results are then used in Section 3, where, as an illustration, the solvable Lie algebras with no more than five ideals are classified completely when the base field is either an algebraically closed field or the real one. The methods of proof are entirely elementary and are applications of linear algebra.

Every Lie algebra considered in this paper will be finite-dimensional over a field F of characteristic zero. Rad(L) [Nil(L)] denotes the largest solvable [nilpotent] ideal of L. The Jacobson radical, $J(L)$, is the intersection of all maximal ideals of L. We denote the terms of the lower central series (l.c.s.) of L by $L = L^1$ and $L^i = [L, L^{i-1}]$ for $i > 1$. The center of L is denoted by $Z(L)$. We define the upper central series (u.c.s.) of L by letting $Z_0(L) = 0$ and $Z_i(L)$ be the ideal of L such that $Z(L/Z_{i-1}(L)) = Z_i(L)/Z_{i-1}(L)$ for $i \geqslant 1$. We shall say that L has nilpotency index n if $L^n = 0$ but $L^{n-1} \neq 0$. Notice that n is the nilpotency index of L if and only if $Z_{n-1}(L) = L$. The symbol $\Im(L)$ denotes the lattice of all ideals of L. Algebra direct sums are denoted by $\oplus$, whereas direct sums of vector-space structures are denoted by $\dot{+}$.

2. BASIC STRUCTURE

LEMMA 2.1. *Let L be a Lie algebra such that $\Im(L)$ is an n-element lattice with $1 \leqslant n \leqslant 5$. Then, $\Im(L)$ is one of the lattices shown in Figure 2.*

Proof. Notice that $\Im(L)$ is as in I if and only if $L = 0$. Assume then $L \neq 0$. It is clear that 0 (L) is an ideal of L with the following property: If P is an ideal of L, then $0 \leqslant P$ ($P \leqslant L$). Thus, $\Im(L)$ has at least two elements. If $n = 2$ or 3, it is immediate that $\Im(L)$ is as in II or III. Suppose now $n = 4$, and let P be a minimal ideal of L. If P is the unique minimal ideal, it follows that $\Im(L)$ is as in IV(a). Otherwise, there exists a minimal ideal Q different from P. Then, $0, P, Q, L$ are the elements of $\Im(L)$. Moreover, $P \cap Q = 0$ and $P + Q = L$. Therefore, $\Im(L)$ is as in IV(b). Finally consider

$n = 5$. In that case, L has at most three minimal ideals. Assume there are three. Write them as P, Q, R. We have that $P \cap Q = P \cap R = R \cap Q = 0$ and $P + Q = P + R = R + Q = L$. Then, as $[P, Q] = [P, R] = [Q, R] = 0$, we deduce that L is abelian of dimensionality 2, which is a contradiction. Therefore, L has at most two minimal ideals. If there is only one, L is as in V(a) or V(c). Otherwise, denote by P, Q the minimal ideals of L, and let K be the fifth ideal. As K is not minimal, we can suppose $P \leqslant K$. If Q is not contained in K, it follows that $L = P + Q$ and therefore $K = P$, which is a contradiction. We conclude that $P + Q = K$, and this yields $\mathfrak{I}(L)$ is as in V(b). ■

In the terminology of lattice theory, a lattice $\mathscr{L}$ which is totally ordered by set inclusion is said to be a chain. Then the lattices represented by the diagrams I, II, III, IV(a), V(a) in Figure 2 are n-element chains with $n = 1$, 2, 3, 4, 5 respectively. The other lattices will be denoted by [Figure 2, $*$], where $* =$ IV(b), V(b), V(c). In [1], we studied the problem of determining the structure of a Lie algebra L for which $\mathfrak{I}(L)$ is a chain. We were able to classify completely the supersolvable Lie algebras in this class. In the general case, we only obtained the classification of those Lie algebras for which $\mathfrak{I}(L)$ is a 1-, 2-, or 3-element chain. Now, from Theorem 2.3, we shall get the complete structure of a Lie algebra L in which $\mathfrak{I}(L)$ is a 4- or 5-element chain. The following lemma is obtained in [1]. Its proof is presented here for completeness.

LEMMA 2.2 [1]. *Let L be a Lie algebra. Suppose* $\dim L > 1$ *and L is not simple. Then the following properties of L are equivalent*:

(i) *L has a unique maximal ideal.*

(ii) $L = \mathrm{Nil}(L) \dotplus S$, *where S is* $1-$ *dimensional or simple and* $\mathrm{ad}_L\, S$ *acts nontrivially on every* $\mathrm{ad}_L\, S$*-invariant subspace of* $\mathrm{Nil}(L)/\mathrm{Nil}(L)^2$.

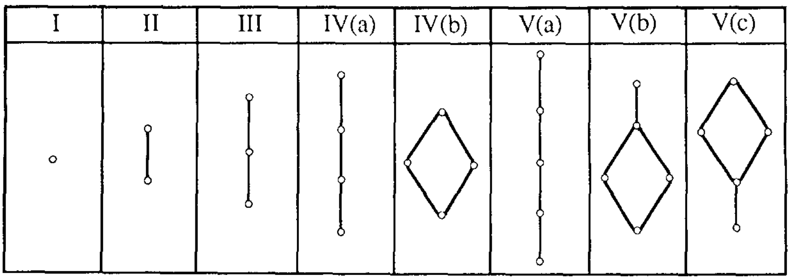

FIG. 2.

Proof. (i) $\Rightarrow$ (ii): We have that $J(L)$ is the unique maximal ideal of L. Then $L/J(L)$ is one-dimensional or simple. From Theorem 3.1 of [6], $J(L) = [L, \mathrm{Rad}(L)] \leqslant \mathrm{Nil}(L)$. We claim that $J(L) = \mathrm{Nil}(L)$. If $L/J(L)$ is simple, it is clear that $J(L) = \mathrm{Nil}(L)$. Assume then $\dim L/J(L) = 1$. If $J(L) \neq \mathrm{Nil}(L)$, we conclude that L is nilpotent. Then $J(L) = L^2$, and L^2 is a nonzero ideal because $\dim L > 1$. Now consider L/L^3. As $\dim L/L^2 = 1$, we can write $L/L^3 = L^2/L^3 \dot{+} ((y))$. Since $L^2/L^3 \leqslant Z(L/L^3)$, we get $L^2 = L^3$, which is a contradiction because L is nilpotent. Thus $J(L) = \mathrm{Nil}(L)$. Therefore $L = \mathrm{Nil}(L) \dot{+} S$, where S is one-dimensional or simple (the decomposition in the case S simple follows from the Levi theorem). Notice that $\mathrm{Nil}(L/\mathrm{Nil}(L)^2) = \mathrm{Nil}(L)/\mathrm{Nil}(L)^2 = J(L/\mathrm{Nil}(L)^2)$ (see [7, Proposition 1.4]). Assume $\mathrm{Nil}(L)^2 = 0$. Then $L = \mathrm{Nil}(L) \dot{+} S$ with $S = ((x))$ or S simple and $\mathrm{Nil}(L)$ abelian. We have the following: If $S = ((x))$, then $\mathrm{Nil}(L) = J(L) = L^2 = [\mathrm{Nil}(L), x]$ and therefore $\mathrm{ad}_L\, x|_{\mathrm{Nil}(L)}$ is nonsingular. If S is simple, $\mathrm{Nil}(L) = J(L) = [L, \mathrm{Rad}(L)] = [S, \mathrm{Nil}(L)]$. In that case, let A be an $\mathrm{ad}_L S$-invariant subspace of $\mathrm{Nil}(L)$. As $\mathrm{Nil}(L)$ is $\mathrm{ad}_L S$-completely reducible (see [3, p. 79]), there exists an $\mathrm{ad}_L S$-invariant subspace B such that $\mathrm{Nil}(L) = A \oplus B$. Then $[A \oplus B, S] = [A, S] \oplus [B, S] = A \oplus B$, and this yields $[A, S] = A$. Therefore, (ii) follows.

(ii) $\Rightarrow$ (i): Write $N = \mathrm{Nil}(L)$. It is easily checked that $N = N^2 + [N, S]$. Then $N \leqslant J(L) = [L, \mathrm{Rad}(L)] \leqslant N$. Thus N is the unique maximal ideal of L. ∎

THEOREM 2.3. *Let L be a Lie algebra. Then L has at most five ideals if and only if one of the following holds:*

(i) $L = 0$.

(ii) *L is either one-dimensional or simple.*

(iii) *$L = N \dot{+} ((x))$, where N is a nonzero abelian ideal, $\mathrm{ad}_L\, x|_N$ is nonsingular and cyclic, and its minimal polynomial has the form $\pi(x)^n$, where $\pi(X)$ is irreducible and $1 \leqslant n \leqslant 3$.*

(iv) *$L = N \dot{+} ((x))$, where N is a nilpotent ideal of nilpotency index three, $\mathrm{ad}_L\, x|_{N/N^2}, \mathrm{ad}_L\, x|_{N^2}$ are cyclic, and their minimal polynomials have the form $\pi(X)^n$, $\mu(X)^m$ respectively, where:*

(a) *$\pi(X)$, $\mu(X)$ are irreducible and $\pi(X) \neq X$.*
(b) *(n, m) is one of the following pairs: $(1,1)$, $(1,2)$, or $(2,1)$. Moreover, if $(n, m) = (2,1)$ and $Z(N) \neq N^2$, then $\pi(X) = \mu(X)$ and the minimal polynomial of $\mathrm{ad}_L\, x|_{Z(N)}$ is $\pi(X)^2$.*

(v) *$L = N \dot{+} ((x))$, where N is a nilpotent ideal of nilpotency index four, $\mathrm{ad}_L\, x|_{N/N^2}, \mathrm{ad}_L\, x|_{N^2/N^3}, \mathrm{ad}_L\, x|_{N^3}$ are cyclic with irreducible minimal polynomials, and $\mathrm{ad}_L\, x|_{N/N^2}$ is nonsingular.*

(vi) $L = N \dot{+} S$, *where* N *is a nonzero nilpotent ideal of nilpotency index* n *with* $2 \leqslant n \leqslant 4$, N/N^2 *is a faithful* $\mathrm{ad}_L\, S$*-module, and* N^i/N^{i+1} *is an irreducible* $\mathrm{ad}_L\, S$*-module for* $1 \leqslant i \leqslant n - 1$.

(vii) $L = ((x)) \oplus S$, *where* S *is a simple ideal and* $((x))$ *is the center.*

(viii) $L = S_1 \oplus S_2$, *where* S_1, S_2 *are simple ideals.*

(ix) $L = N \dot{+} ((x))$, *where* N *is a nonzero abelian ideal,* $\mathrm{ad}_L\, x|_N$ *is nonsingular and cyclic, and its minimal polynomial is a product of two distinct irreducibles.*

(x) $L = N \dot{+} S$, *where* N *is an abelian ideal,* S *is a simple subalgebra, and* $N = N_1 \oplus N_2$ *with* N_1, N_2 *irreducible, faithful, and nonisomorphic* $\mathrm{ad}_L\, S$*-modules.*

(xi) $L = N \dot{+} S$, *where* N *is an abelian ideal,* S *is an algebra of the type described in* (vii) *or* (viii), *and* N *is an irreducible and faithful* $\mathrm{ad}_L\, S$*-module.*

Moreover, if L *is as in* (xi) *with* $S = ((x)) \oplus S_1$, *the minimal polynomial of* $\mathrm{ad}_L\, x|_N$ *is irreducible. Thus if* $\mathrm{ad}_L\, x|_N$ *is split, then the transformation* $\mathrm{ad}_L\, x$ *is scalar in* N *and* N *must be an irreducible* $\mathrm{ad}_L\, S_1$*-module.*

Proof. From Lemma 2.1, $\mathfrak{I}(L)$ is one of the lattice diagrams in Figure 2. We study each diagram separately.

Case 1: $\mathfrak{I}(L)$ *is an* n*-element chain,* $1 \leqslant n \leqslant 5$. If $n = 1$ or 2 it is immediate that L is as in (i) or (ii). Suppose then $n \geqslant 3$. As L has a unique maximal ideal, from Lemma 2.2 we have that $L = N \dot{+} S$, where $N = \mathrm{Nil}(L) \neq 0$, S is a simple subalgebra or $S = ((x))$, and $\mathrm{ad}_L\, S$ acts nontrivially on every $\mathrm{ad}_L\, S$-invariant subspace of N/N^2. Let k be the nilpotency index of N. Then $k \leqslant 4$, because the terms of the l.c.s. of N are ideals of L. Assume first S is simple, and consider N^i/N^{i+1} for $1 \leqslant i \leqslant k - 1$. We can write $N^i/N^{i+1} = A_1/N^{i+1} \oplus \ldots \oplus A_m/N^{i+1}$, where each A_j/N^{i+1} is an irreducible $\mathrm{ad}_L\, S$-module [3, p. 79]. Then A_j is an ideal of L for $1 \leqslant j \leqslant m$. As $\mathfrak{I}(L)$ is a chain, we conclude that $m = 1$ and therefore L is as in (vi). Now, suppose $S = ((x))$. Consider N^i/N^{i+1} for $1 \leqslant i \leqslant k - 1$. Notice that every $\mathrm{ad}_L\, x$-stable subspace of N^i/N^{i+1} is an ideal of L. As $\mathfrak{I}(L)$ is a chain, it follows that $\mathrm{ad}_L\, x|_{N^i/N^{i+1}}$ is an indecomposable transformation. Therefore, $\mathrm{ad}_L\, x|_{N^i/N^{i+1}}$ is cyclic and its minimal polynomial is a power of an irreducible [4, p. 129]. If N is abelian, we obtain that L is as in (iii). Assume then $k = 3$. We have that $0, N^2, N, L \in \mathfrak{I}(L)$. Thus $\mathfrak{I}(L)$ has 4 or 5 elements. If $\mathfrak{I}(L)$ is a 4-element chain, it follows that $0 < N^2 < N < L$ is the chain of ideals of L. Therefore L is as in (iv) with $(n, m) = (1, 1)$. If $\mathfrak{I}(L)$ is a 5-element chain, there exists an ideal K such that either $0 < K <$

N^2 or $N^2 < K < N$. In the first case, L is as in (iv) with $(n, m) = (1, 2)$. In the second one, we conclude that L is as in (iv) with $(n, m) = (2, 1)$. In this last case, if $Z(N) \neq N^2$, it follows that $Z(N) = K$. As every $\mathrm{ad}_L\, x$-stable subspace of $Z(N)$ is an ideal of L, $((x))$ acts cyclically on $Z(N)$ with minimal polynomial $\delta(X)^2$, where $\delta(X)$ is irreducible. Denote by $\pi(X)^2$ and $\mu(X)$ the minimal polynomials of $\mathrm{ad}_L\, x$ on N/N^2 and N^2 respectively. Notice that $N^2 < Z(N) < N$. Thus, $\delta(X) = \pi(X) = \mu(X)$, which proves the last assertion of (iv)(b). Finally, suppose $k = 4$. Then $0 < N^3 < N^2 < N < L$ is the chain of ideals of L, and this yields that L is as in (v).

Case 2: $\Im(L)$ is as in [Figure 2, IV(b)]. Then, $L = A \oplus B$, where A, B are the only two minimal ideals of L. If both of them are abelian, we conclude that L is abelian of dimensionality 2, which is a contradiction. Therefore, L is as in (vii) or (viii).

Case 3: $\Im(L)$ is as in [Figure 2, V(b)]. From Lemma 2.2, we have that $L = N \dot{+} S$, where $N = \mathrm{Nil}(L) \neq 0$, S is one-dimensional or simple, and $\mathrm{ad}_L\, S$ acts nontrivially on every $\mathrm{ad}_L\, S$-invariant subspace of N/N^2. Moreover, $N = A \oplus B$, where A, B are the only two minimal ideals of L. It follows that N is abelian. Notice that $\mathrm{ad}_L\, S$ acts irreducibly on A and B because of their minimality. Suppose first $S = ((x))$. Then $\mathrm{ad}_L\, x|_A$ and $\mathrm{ad}_L\, x|_B$ are cyclic with prime minimal polynomials $\pi(X)$, $\mu(X)$ respectively (see [4, p. 128]). If $\pi(X) = \mu(X)$, we can take $\{a_i\}_{1 \leqslant i \leqslant n}, \{b_i\}_{1 \leqslant i \leqslant n}$ as a basis for A and B with respect to which $\mathrm{ad}_L\, x|_A$, $\mathrm{ad}_L\, x|_B$ are represented by the same Jordan canonical matrix (see [4, p. 70]). Then the subspace $P = ((a_i + b_i : 1 \leqslant i \leqslant n))$ is an ideal of L different from A and B, which is a contradiction. Therefore $\pi(X) \neq \mu(X)$, and this yields L is as in (ix). Now, assume S is simple. If A, B are isomorphic $\mathrm{ad}_L\, S$-modules, we can take $\{a_i\}_{1 \leqslant i \leqslant n}, \{b_i\}_{1 \leqslant i \leqslant n}$ basis of A and B with respect to which the action of $\mathrm{ad}_L\, S$ is the same. Then $P = ((a_i + b_i : 1 \leqslant i \leqslant n))$ is a minimal ideal of L different from A, B, which is a contradiction. Therefore, L is as in (x).

Case 4: $\Im(L)$ is as in [Figure 2, V(c).] We have that the Jacobson radical, $J(L)$, is unique minimal ideal of L, and $L/J(L)$ is as in (vii) or (viii). As $J(L)$ is nilpotent (see [6, Corollary 3.1]), we conclude that $J(L)$ is abelian. Suppose first $L/J(L)$ is as in (viii). Then $J(L) = \mathrm{Rad}(L)$ and, from the Levi theorem, $L = J(L) \dot{+} S_1$, where S_1 is a direct sum of two simple subalgebras. Now, assume $L/J(L)$ is as in (vii). Then, $J(L)$ is of codimension 1 in $\mathrm{Rad}(L)$ and $L = \mathrm{Rad}(L) \dot{+} S_2$ where S_2 is a simple subalgebra. As $J(L)$ and $\mathrm{Rad}(L)$ are $\mathrm{ad}_L\, S_2$-modules, we have that $\mathrm{Rad}(L) = J(L) \dot{+} B$, where B is a one-dimensional $\mathrm{ad}_L\, S_2$-module. Therefore $B = ((x))$ and $[x, S_2] = 0$ (see [5, p. 28]). Consequently, we can decompose L as $J(L) \dot{+} S$, where S is a subalgebra as in (vii) or (viii). Notice that $\mathrm{ad}_L\, S$ acts faithfully and irreducibly on $J(L)$ in both cases, because $J(L)$ is the unique minimal ideal of L. Then L is as in (xi).

Now we shall prove the converse. It is immediate if L is as in (i), (ii), (vii), or (viii). Notice that if L is as in (iii), (iv), (v), (vi), or (ix), then from Lemma 2.2 N is the unique maximal ideal of L. Suppose L is as in (iii) or (ix). Then the proper ideals of L are the $\mathrm{ad}_L\, x$-invariant subspaces of N. Therefore, if L is as in (iii), we conclude that $\Im(L)$ is one of the following chains:

$$0 < \mathrm{Ker}\big(\pi(\mathrm{ad}_L\, x|_N)\big) = N < L \qquad \text{for} \quad n = 1,$$

$$0 < \mathrm{Ker}\big(\pi(\mathrm{ad}_L\, x|_N)\big) < \mathrm{Ker}\big(\pi^2(\mathrm{ad}_L\, x|_N)\big) = N < L \qquad \text{for} \quad n = 2,$$

$$0 < \mathrm{Ker}\big(\pi(\mathrm{ad}_L\, x|_N)\big) < \mathrm{Ker}\big(\pi^2(\mathrm{ad}_L\, x|_N)\big)$$
$$< \mathrm{Ker}\big(\pi^3(\mathrm{ad}_L\, x|_N)\big) = N < L \qquad \text{for} \quad n = 3,$$

where $\mathrm{Ker}(\pi^i(\mathrm{ad}_L\, x|_N)) = \{a \in N : a\pi^i(\mathrm{ad}_L\, x|_N)) = 0\}$. If L is as in (ix) and we denote by $\pi(X)\mu(X)$ the minimal polynomial of $\mathrm{ad}_L\, x|_N$, we obtain that 0, $\mathrm{Ker}(\pi(\mathrm{ad}_L\, x|_N))$, $\mathrm{Ker}(\mu(\mathrm{ad}_L\, x|_N))$, N, and L are the elements of $\Im(L)$.

Now, let L be as in (iv) with $(n, m) = (1, 1)$ or $(1, 2)$. Pick an ideal P of L such that $N^2 \leqslant P \leqslant N$. Since $((x))$ acts irreducibly on N/N^2 in both cases, $P = N^2$ or $P = N$. It follows that $Z(N) = N^2$. Next, let K be a minimal ideal of L. From [7, Propositions 3.7], we have that $K \leqslant Z(N) = N^2$. If $(n, m) = (1, 1)$, we conclude that $K = N^2$ is the unique minimal ideal of L. Thus $\Im(L)$ is the 4-element chain $0 < N^2 < N < L$. If $(n, m) = (1, 2)$, then as $((x))$ acts cyclically on N^2, we obtain that $K = \mathrm{Ker}(\pi(\mathrm{ad}_L\, x|_{N^2}))$. Then, $\mathrm{Ker}(\pi(\mathrm{ad}_L\, x|_{N^2}))$ is the unique minimal ideal of L. Moreover, $L/\mathrm{Ker}(\pi(\mathrm{ad}_L\, x|_{N^2}))$ is as in (iv) with $(n, m) = (1, 1)$. It follows that $\Im(L)$ is the 5-element chain $0 < \mathrm{Ker}(\pi(\mathrm{ad}_L\, x|_{N^2})) < N^2 < N < L$.

Next assume L is as in (iv) with $(n, m) = (2, 1)$. Notice that N^2 is a minimal ideal. If $Z(N) = N^2$ as in the above paragraph, it is easily checked that N^2 is the unique minimal ideal of L. Now, L/N^2 is as in (iii) with $n = 2$. Thus, $\Im(L)$ is a 5-element chain. If $Z(N) \neq N^2$, then from (iv) (b), $((x))$ acts cyclically on $Z(N)$. It follows that $\mathrm{Ker}(\pi(\mathrm{ad}_L\, x|_{Z(N)}))$ and $\mathrm{Ker}(\pi^2(\mathrm{ad}_L\, x|_{Z(N)})) = Z(N)$ are the only two $\mathrm{ad}_L\, x$-invariant subspaces of $Z(N)$. Then we deduce that $N^2 = \mathrm{Ker}(\pi(\mathrm{ad}_L\, x|_{Z(N)}))$, which is the unique minimal ideal of L. Consequently, as in the above case, we obtain that $\Im(L)$ is a 5-element chain. Now, let L be as in (v). We have that $N^3 \leqslant Z(N)$. Suppose $N^3 \neq Z(N)$. The Lie algebra L/N^3 is as in (iv) with $(n, m) = (1, 1)$. Then $0 < N^2/N^3 < N/N^3 < L/N^3$ is the lattice of ideals of L/N^3. It follows that $Z(N) = N$ or N^2, which is a contradiction. Therefore $Z(N) =$

N^3, and this yields that N^3 is the unique minimal ideal for L. Then, from (iv), we conclude that L is a 5-element chain. If L is as in (vi), it is easily checked that the l.c.s. of N coincides with the u.c.s. Consequently, $\Im(L)$ is the $(n+1)$-element chain $0 < N^{n-1} < \cdots < N < L$, where $n = 2, 3, 4$.

Now, let L be as in (x). Take P a minimal abelian ideal of L. Then $P \leqslant N$, because $N = \mathrm{Nil}(L)$. As $\mathrm{ad}_L\, S$ is completely reducible on N (see [3, p. 79]), there exists an $\mathrm{ad}_L\, S$-module C such that $N = P \oplus C$. From the Jordan-Hölder theorem, we can suppose that P and N_1 are isomorphic $\mathrm{ad}_L\, S$-modules. Then $P \cap N_2 = 0$. If $P \neq N_1$, we obtain that $N = P \oplus N_1 = P \oplus N_2$, and this yields that N_1, N_2 are isomorphic $\mathrm{ad}_L\, S$-modules, which is a contradiction. It follows that N_1, N_2 are the unique $\mathrm{ad}_L\, S$-invariant subspaces of N. Thus, from Lemma 2.2, N is the unique maximal ideal of L. Consequently, 0, N_1, N_2, N, and L are the ideals of L.

Finally, assume L is as in (xi). Suppose first L/N is as in (viii). It follows that $N = \mathrm{Nil}(L)$. Let P be a minimal ideal of L. Then P is either abelian or simple. If P is simple, we deduce that $P \leqslant S$ (see [3, p. 92]). As $[P, N] = 0$, it follows that $\mathrm{ad}_L S$ is not faithful on N, which is a contradiction. Thus P is abelian, and therefore $P = N$ because of the minimality of N. Now, as $S = S_1 \oplus S_2$, we conclude that 0, N, $N \dotplus S_1$, $N \dotplus S_2$, L are the ideals of L. Next, let $L = N \dotplus ((x)) \oplus S$. We claim that $N = \mathrm{Nil}(L)$. Notice that $\mathrm{Rad}(L) = N \dotplus ((x))$. Thus we need only prove that $\mathrm{ad}_L\, x|_N$ is not nilpotent. If that is not the case, from Engel's theorem (see [3, p. 36]) the minimal polynomial of $\mathrm{ad}_L\, x|_N$ is of the form X^n. As $\mathrm{ad}_L[((x)) \oplus S]$ is completely reducible on N, it follows that $\mathrm{ad}_L\, x|_N$ is semisimple (see [3, p. 81]). Consequently, $[N, x] = 0$, which is a contradiction. Now, as in the above case, it is easily checked that N is the unique minimal ideal of L. Then we conclude that 0, N, $N \dotplus ((x))$, $N \dotplus S$, and L are the ideals of L.

To prove the last assertion of the theorem, consider $L = N \dotplus ((x)) \oplus S_1$. Notice that $\mathrm{ad}_L\, x|_N$ is semisimple because $\mathrm{ad}_L[((x)) \oplus S_1]$ is completely reducible on N (see [3, p. 81]). Then we can write its minimal polynomial as $\pi_1(X) \cdots \pi_n(X)$ with $\pi_i(X)$ distinct irreducibles for $1 \leqslant i \leqslant n$. Consequently, we can decompose N as $N_{\pi_1} \dotplus \cdots \dotplus N_{\pi_n}$, where $N_{\pi_i} = \{a \in N : a\pi_i(\mathrm{ad}_L\, x) = 0\}$. On the other hand, $[\mathrm{ad}_L\, x, \mathrm{ad}_L\, s] = 0$ for every $s \in S_1$. Then from [3, p. 40] we obtain that N_{π_i} is $\mathrm{ad}_L\, S_1$-invariant for $1 \leqslant i \leqslant n$, and this yields that each N_{π_i} is an ideal of L. As N is a minimal ideal, we conclude that $n = 1$. Now the last part of the assertion is immediate. The proof is complete. ∎

REMARK 2.4. Notice that every Lie algebra L listed in Theorem 2.3 determines $\Im(L)$ uniquely as one of the lattices in Figure 2. Moreover, given a lattice $\mathscr{L}$ of the type described in Figure 2, if there exists a Lie algebra L

for which $\mathcal{L}$ is its lattice of ideals, $\mathcal{L}$ determines the structure of L (not uniquely). This allows us to deal with the existence problem that will be our next task.

3. THE EXISTENCE PROBLEM

The results of the last section reveal the structure of the Lie algebras which have at most five ideals. But it remains to be shown that these structures are possible. The existence problem for Lie algebras of types (i), (ii), (iii), (vii), (viii), (ix), (x) in Theorem 2.3 is readily solved, and in each case it is easy to obtain a basis and give the corresponding multiplication table. If the base field is algebraically closed, then case (xi) is easy too. The existence problem for the rest will be solved by the two following corollaries and the final examples in Example 3.4.

LEMMA 3.1. *Let L be a Lie algebra direct sum of a nilpotent ideal N and a one-dimensional subalgebra $((x))$. Then $\mathrm{ad}_L\, x|_N$ is split if and only if $\mathrm{ad}_L\, x|_{N/N^2}$ is split.*

Proof. The "only if" is immediate. We shall prove the "if." Let S be the largest subspace of N such that $\mathrm{ad}_L\, x|_S$ is split. Notice that S is a subalgebra (see [3, p. 64]). Then $\mathrm{ad}_L\, x|_{N/(N^2+S)}$ does not have eigenvalues in the field. It follows that $N = N^2 + S$. Let M be a maximal subalgebra such that $S \dotplus (x) \leqslant M$. By Theorem 6.5 of [8], $N^2 \leqslant M$. Then $M = L$, which is a contradiction. Therefore $S = N$, and this yields that $\mathrm{ad}_L\, x|_N$ is split. ∎

COROLLARY 3.2. *Let L be a solvable Lie algebra over an algebraically closed field of characteristic zero. Then, L has at most five ideals if and only if one of the following holds (only nonzero products are given):*

(1) $L = 0$.

(2) *L is one-dimensional.*

(3) *L is the $(n+1)$-dimensional Lie algebra, $1 \leqslant n \leqslant 3$, with basis $\{a_1, \ldots, a_n, y\}$ and products $[a_i, y] = a_i + a_{i+1}$ for $1 \leqslant i \leqslant n-1$, $[a_n, y] = a_n$.*

(4) *L is the four-dimensional Lie algebra with basis $\{a_1, a_2, a_3, y\}$ and products $[a_1, a_2] = a_3$, $[a_1, y] = a_1 + a_2$, $[a_2, y] = a_2$, $[a_3, y] = 2a_3$.*

(5) *L is the three-dimensional Lie algebra with basis $\{a_1, a_2, y\}$ and products $[a_1, y] = a_1$, $[a_2, y] = \alpha a_2$, where $\alpha \neq 0, 1$.*

Moreover, two Lie algebras as in (5) *corresponding to scalars* α, β *are isomorphic if and only if* $\alpha = \beta$ *or* $\alpha = 1/\beta$.

Proof. Suppose $\dim L > 1$ and L has no more than five ideals, so that L has the structure described in (iii), (iv), (v), or (ix) of Theorem 2.3. Then L is a direct sum of a nonzero nilpotent ideal N and a one-dimensional subalgebra $((x))$. We study the different cases separately.

(a) Let L be as in Theorem 2.3 (iii). It follows that N is an n-dimensional abelian ideal and $\mathrm{ad}_L\, x|_N$ has minimal polynomial $(X - \alpha)^n$, where $1 \leqslant n \leqslant 3$, $\alpha \neq 0$. Take $y = \mu x$ with $\mu = 1/\alpha$, and pick a basis $\{a_i\}_{1 \leqslant i \leqslant n}$ for N with respect to which $\mathrm{ad}_L\, y|_N$ is represented by a matrix in classical canonical form (see [4, p. 73]). Then $\{a_i, y\}_{1 \leqslant i \leqslant n}$ is a basis for L as in 3).

(b) Let L be as in Theorem 2.3 (iv). As $\dim N/N^2 > 1$, there is only one possibility, which is that $\dim N = 3$, $\dim N^2 = 1$, and $((x))$ acts on N/N^2 and N^2 with minimal polynomials $(X - \alpha)^2$, $\alpha \neq 0$, $X - \beta$ respectively. Now we consider the subspace $N_\alpha = \{z \in N : z(\mathrm{ad}_L\, x - \alpha I)^n = 0\}$. We have that $N = N_\alpha + N^2$. As $N^2 \leqslant Z(N)$, we have $0 \neq N^2 = [N_\alpha, N_\alpha] \leqslant N_{2\alpha}$ (see [3, p. 64]), and this yields $\beta = 2\alpha$. Consequently, we can decompose $N = N_\alpha + N_{2\alpha}$, where $N_\alpha = \{z \in N : z(\mathrm{ad}_L\, x - \alpha I)^2 = 0\}$ and $N_{2\alpha} = \{a \in N : [a, x] = 2\alpha a\} = N^2$. Now, we can take a basis a, b, c, for N such that $[a, x] = \alpha a + b$, $[b, x] = \alpha b$, $[c, x] = 2\alpha c$. Notice that $[a, b] = \lambda c$ for some $\lambda \neq 0$. Then $a, \mu b, \lambda\mu c, \mu x$, where $\mu = 1/\alpha$, is a basis for L as in (4).

(c) From [3, p.11], there exist only two nilpotent Lie algebras of dimensionality 3: the 3-dimensional abelian Lie algebra and the Lie algebra $M = ((x, y, z))$ with products $[x, y] = z$, $[z, x] = [z, y] = 0$. Hence there are no Lie algebras as in Theorem 2.3 (v).

(d) Let L be as in Theorem 2.3 (ix). Then L has a basis a_1, a_2, x such that $[a_1, x] = \delta a_1$ and $[a_2, x] = \beta a_2$, where $\delta, \beta \neq 0$ and $\delta \neq \beta$. It follows that $a_1, a_2, \mu x$, where $\mu = 1/\delta$, is a basis for L as in (5) with $\alpha = \beta/\delta$.

The last assertion is easily checked. The converse follows from Theorem 2.3. ∎

COROLLARY 3.3. *Let L be a solvable Lie algebra over the real field. Then L has at most five ideals if and only if one of the following holds* (*only nonzero products are given*):

(1) *L is one of the Lie algebras listed in Corollary* 3.2.

(2) *L is the* $(2n + 1)$*-dimensional Lie algebra with basis* $\{a_1, f_1, \ldots, a_n, f_n, x\}$, $1 \leqslant n \leqslant 3$, *and products* $[a_i, x] = f_i$, $[f_i, x] = -a_i + \alpha f_i + a_{i+1}$ *for* $1 \leqslant i \leqslant n - 1$, $[a_n, x] = f_n$, $[f_n, x] = -a_n + \alpha f_n$ *with* $0 \leqslant \alpha < 2$.

(3) *L is the 4-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, x\}$ and products $[a_1, x] = a_2$, $[a_2, x] = -a_1 + \alpha a_2$, $[a_3, x] = \alpha a_3$, $[a_1, a_2] = a_3$ *with* $0 \leqslant \alpha < 2$.

(4) *L is the 6-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, a_5, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1 + \alpha a_2 + a_3$, $[a_3, x] = a_4$, $[a_4, x] = -a_3 + \alpha a_4$, $[a_5, x] = \alpha a_5$, $[a_1, a_4] = -a_5$, $[a_2, a_3] = a_5$ *with* $0 \leqslant \alpha < 2$.

(5) *L is the 7-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, a_5, a_6, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1 + \alpha a_2 + a_3$, $[a_3, x] = a_4$, $[a_4, x] = -a_3 + \alpha a_4$, $[a_5, x] = a_6$, $[a_6, x] = -4a_5 + 2\alpha a_6$, $[a_1, a_3] = a_5$, $[a_2, a_3] = [a_1, a_4] = \frac{1}{2}a_6$, $[a_2, a_4] = -a_5 + (\alpha/2)a_6$, $[a_1, a_2] = -[\alpha/(\alpha^2 - 4)]a_5 + [1/(\alpha^2 - 4)]a_6$ *with* $0 \leqslant \alpha < 2$.

(6) *L is the 6-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, a_5, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1 + \alpha a_2$, $[a_3, x] = \alpha a_3$, $[a_4, x] = a_5$, $[a_5, x] = -(2\alpha^2 + 1)a_4 + 3\alpha a_5$, $[a_1, a_2] = a_3$, $[a_1, a_3] = a_4$, $[a_2, a_3] = -\alpha a_4 + a_5$ *with* $0 \leqslant \alpha < 2$.

(7) *L is the 6-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, a_5, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1 + a_4$, $[a_3, x] = 0$, $[a_4, x] = a_5$, $[a_5, x] = -a_4$, $[a_1, a_2] = a_3$, $[a_1, a_3] = a_4$, $[a_2, a_3] = a_5$.

(8) *L is the 6-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, a_5, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1 + a_4$, $[a_3, x] = 0$, $[a_4, x] = a_5$, $[a_5, x] = -a_4$, $[a_1, a_2] = a_3$, $[a_1, a_3] = a_5$, $[a_2, a_3] = -a_4$.

(9) *L is the 4-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, x\}$ *and products* $[a_1, x] = a_1$, $[a_2, x] = a_3$, $[a_3, x] = \beta a_2 + \alpha a_3$ *with* $\alpha^2 + 4\beta < 0$.

(10) *L is the 5-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1 + \alpha a_2$, $[a_3, x] = a_4$, $[a_4, x] = \beta a_3 + \mu a_4$ *with* $0 < \alpha < 2$, $-1 < \beta < 0$, *and* $\mu^2 + 4\beta < 0$.

(11) *L is the 5-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1$, $[a_3, x] = a_4$, $[a_4, x] = -a_3 + \mu a_4$ *with* $-2 < \mu < 2$, $\mu \neq 0$ *or* α, *and* $0 < \alpha < 2$. *Moreover, two Lie algebras of this family corresponding to scalars* (α, μ), (α', μ') *are isomorphic if and only if* $\alpha' = -\mu$ *and* $\mu' = -\alpha$.

(12) *L is the 5-dimensional Lie algebra with basis* $\{a_1, a_2, a_3, a_4, x\}$ *and products* $[a_1, x] = a_2$, $[a_2, x] = -a_1$, $[a_3, x] = a_4$, $[a_4, x] = \beta a_3 + \mu a_4$ *with* $0 \leqslant \mu$, $-1 \leqslant \beta < 0$, $\mu^2 + 4\beta < 0$, *and* $(\beta, \mu) \neq (-1, 0)$.

No two algebras described above (including different members of the families) are isomorphic.

Proof. Suppose dim $L > 1$ and L has no more than five ideals, so that L has the structure described in (iii), (iv), (v), or (ix) of Theorem 2.3. Then L is a direct sum of a nilpotent ideal N and a one-dimensional subalgebra $((x))$. Take notice that the irreducible polynomials over the real field are linear or quadratic. If $\mathrm{ad}_L\, x|_{N/N^2}$ is split, from Lemma 3.1 we conclude that L is one of the Lie algebras listed in Corollary 3.2. Then we can assume $\mathrm{ad}_L\, x|_{N/N^2}$ is not split. We study the different cases separately.

(a) Let L be as in Theorem 2.3 (iii). Then the minimal polynomial of $\mathrm{ad}_L\, x|_N$ has the form $(X^2 - c_1 X - c_0)^n$, where $1 \leqslant n \leqslant 3$, $c_1^2 + 4c_0 < 0$, and N is a $2n$-dimensional abelian ideal. Take $y = \mu x$, where $\mu = 1/\sqrt{-c_0}$. It follows that the minimal polynomial of $\mathrm{ad}_L\, y|_N$ is $(X^2 - \alpha X + 1)^n$ with $\alpha = c_1/\sqrt{-c_0}$. Notice that $-2 < \alpha < 2$. Consequently, we can pick a basis $a_1, f_1, \dots, a_n, f_n$ for N with respect to which $\mathrm{ad}_L\, y|_N$ is represented by a matrix in classical canonical form. Then $a_1, f_1, \dots, a_n, f_n, y$ is a basis for L as in (2). It is easily checked that two Lie algebras of this family corresponding to scalars α, α' are isomorphic if and only if $\alpha' = \pm\alpha$. Therefore, we can take the restriction $0 \leqslant \alpha < 2$.

(b) Let L be as in Theorem 2.3 (iv). Then, $0 \neq N^2 \leqslant Z(N)$ because the nilpotency index of N is three. As L/N^2 is as in Theorem 2.3 (iii) with $n = 1, 2$, from (a) we have $\dim N/N^2 = 2$ or 4, and we can consider without loss of generality that the minimal polynomial of $\mathrm{ad}_L\, x|_{N/N^2}$ is either $X^2 - \alpha X + 1$ or $(X^2 - \alpha X + 1)^2$ respectively with $0 \leqslant \alpha < 2$. Now, let $\mu(X)^m$ be the minimal polynomial of $\mathrm{ad}_L\, x|_{N^2}$. Notice that if $\dim N/N^2 = 2$, then $\dim N^2 = 1$. Therefore $m = 1$, and $\mu(X)$ is a linear polynomial. If $\dim N/N^2 = 4$, from Theorem 2.3 (iv)(b) we conclude that $m = 1$ and $\mu(X)$ is either a linear or a quadratic irreducible polynomial. Consequently, $\mathrm{ad}_L\, x$ acts irreducibly on N^2, and its minimal polynomial has the form either $X - \beta$ or $X^2 - c_1 X - c_0$ where $c_1^2 + 4c_0 < 0$. We claim that $\beta = \alpha$, $c_1 = 2\alpha$, and $c_0 = -4$. Consider the Lie algebra $L_{\mathbb{C}} = L \otimes \mathbb{C}$. Notice that $0 \neq (N_{\mathbb{C}})^2 = (N^2)_{\mathbb{C}} \leqslant Z(N_{\mathbb{C}})$ (see [3, p. 27]) and $\mathrm{ad}_L\, x|_N$, $\mathrm{ad}_{L_{\mathbb{C}}}\, x|_{N_{\mathbb{C}}}$ have the same characteristic polynomial. Denote by $\delta, \bar{\delta} \in \mathbb{C}$ the roots of $X^2 - \alpha X + 1$. Then we can decompose $N_{\mathbb{C}}$ as $(N_{\bar{\delta}} \dotplus N_{\delta}) + (N_{\mathbb{C}})^2$. We have that $\delta + \bar{\delta} = \alpha$; then from [3, p. 64], $0 \neq (N_{\mathbb{C}})^2 \leqslant (N_{\mathbb{C}})_{2\bar{\delta}} \dotplus (N_{\mathbb{C}})_{2\delta} \dotplus (N_{\mathbb{C}})_{\alpha}$, where $(N_{\mathbb{C}})_{\nu} = 0$ if ν is not a root of the characteristic polynomial of $\mathrm{ad}_{L_{\mathbb{C}}}\, x|_{N_{\mathbb{C}}}$. Consequently, either $\beta = \alpha$ or $2\delta, 2\bar{\delta} \in \mathbb{C}$ are the roots of $X^2 - c_1 X - c_0$, and therefore, $c_1 = 2\alpha$ and $c_0 = -4$, which proves our claim. Then, if $\dim N/N^2 = 2$, it follows that $\dim N^2 = 1$ and $X - \alpha$ must be the minimal polynomial of $\mathrm{ad}_L\, x|_{N^2}$. Thus, we can decompose $N = N_{\pi(X)} \dotplus N_{\alpha}$, where $\pi(X) = X^2 - \alpha X + 1$, $N_{\pi(X)} = \{z \in N : z\pi(\mathrm{ad}_L\, x) = 0\}$, and $N_{\alpha} = \{a \in N : [a, x] = \alpha a\}$. Notice that $N^2 = N_{\alpha}$. Now, we can pick a basis a_1, a_2 for $N_{\pi(X)}$ such that $[a_1, x] = a_2$ and $[a_2, x] = -a_1 + \alpha a_2$. Then $N^2 = (([a_1, a_2]))$, and therefore $a_1, a_2, [a_1, a_2], x$ is a basis for L as in (3). If $\dim N/N^2 = 4$, it follows $\dim N^2 = 1$ or 2. Thus, the minimal polynomial of $\mathrm{ad}_L\, x|_{N^2}$ is $X - \alpha$ or $X^2 - 2\alpha X + 4$. In the first case, we can take a basis p_1, q_1, p_2, q_2, z for N such that $[p_1, x] = q_1$, $[p_2, x] = q_2$, $[q_1, x] = -p_1$

$+ \alpha q_1 + p_2$, $[q_2, x] = -p_2 + \alpha q_2$, $[z, x] = \alpha z$. By using the Jacobi identity, we obtain the following identities:

$$
\begin{aligned}
[[p_1, q_1]x] &= \alpha[p_1, q_1] + [p_1, p_2], \\
[[p_1, p_2]x] &= [q_1, p_2] + [p_1, q_2], \\
[[p_1, q_2]x] &= [q_1, q_2] - [p_1, p_2] + \alpha[p_1, q_2], \\
[[q_1, p_2]x] &= -[p_1, p_2] + \alpha[q_1, p_2] + [q_1, q_2], \\
[[q_1, q_2]x] &= -[p_1, q_2] + 2\alpha[q_1, q_2] + [p_2, q_2] - [q_1, p_2], \\
[[p_2, q_2]x] &= \alpha[p_2, q_2].
\end{aligned}
\tag{3.3.1}
$$

From the above identities, as $\mathrm{ad}_L\, x|_{N^2} = I\alpha$, it is easily checked that $[p_1, p_2] = [q_1, q_2] = [p_2, q_2] = 0$ and $[p_1, q_1] = \lambda z$, $[q_1, p_2] = -[p_1, q_2] = \mu z$, where $\lambda, \mu \in \mathbb{R}$. If $\mu = 0$, we get that $p_2, q_2 \in Z(N)$, which contradicts the last assertion of (iv)(b) in Theorem 2.3. Thus, $\mu \neq 0$. Now, write $k = \lambda/\mu$ and take the following basis for N:

$$a_1 = \alpha p_1 - q_1 + \frac{\alpha(\alpha^2 k + 2)}{\alpha^2 + 2} p_2 - \frac{\alpha^2 k + k + 1}{\alpha^2 + 2} q_2,$$

$$a_2 = p_1 + \frac{(\alpha^2 + 1)(k - 1)}{\alpha^2 + 2} p_2 - \frac{\alpha(k - 1)}{\alpha^2 + 2} q_2,$$

$$a_3 = \alpha p_2 - q_2, \quad a_4 = p_2, \quad a_5 = \mu z.$$

It is easily checked that $a_1, a_2, a_3, a_4, a_5, x$ is a basis for L as in (4). Now, assume $\dim N^2 = 2$. Then the minimal polynomial of $\mathrm{ad}_L\, x|_{N^2}$ has the form $X^2 - 2\alpha X + 4$. Write $\pi(X) = X^2 - \alpha X + 1$ and $\mu(X) = X^2 - 2\alpha X + 4$. We can decompose $N = N_{\pi(X)} \dotplus N_{\mu(X)}$. Take a basis p_1, q_1, p_2, q_2 for $N_{\pi(X)}$ such that $[p_1, x] = q_1$, $[q_1, x] = -p_1 + \alpha q_1 + p_2$, $[p_2, x] = q_2$, $[q_2, x] = -p_2 + \alpha q_2$. As $N_{\mu(X)} = N^2 = Z(N)$, we have that $N^2 = [N_{\pi(X)}, N_{\pi(X)}] = \mathbb{R}(([a, b]: a, b \in \{p_1, q_1, p_2, q_2\}))$. Notice that the identities listed in (3.3.1) hold. Consequently, $[p_2, q_2] = 0$. If $[p_1, p_2] = 0$, from (3.3.1), we obtain that $N^2 = 0$ which is a contradiction. Thus $0 \neq [p_1, p_2] \in N^2$. Denote $[p_1, p_2] = z_1$. Write $[z_1, x] = z_2$. We have that z_1, z_2 is a basis for N^2 such that $[z_2, x] = -4z_1 + 2\alpha z_2$. Now, from this last identity and (3.3.1), it is easily checked that $p_1, q_1, p_2, q_2, z_1, z_2, x$ is a basis for L as in (5).

(c) Let L be as in Theorem 2.3 (v). As L/N^3 is as in Theorem 2.3 (iv) with $(n, m) = (1, 1)$, from (b) we conclude that $\dim N/N^2 = 2$ and $\dim N^2/N^3 = 1$. Moreover, we can consider without loss of generality that the minimal polynomial of $\mathrm{ad}_L\, x|_{N/N^3}$ has the form $(X^2 - \alpha X + 1)(X - \alpha)$ with $0 \leqslant \alpha < 2$. Now, by using the same argument as in the above paragraph (b) with the Lie algebra $L_{\mathbb{C}} = L \otimes \mathbb{C}$, it is easily checked that $\mathrm{ad}_L\, x$ must act on N^3 with minimal polynomial either $X - 2\alpha$ or $X^2 - 3\alpha X + (2\alpha^2 + 1)$. Suppose first that the minimal polynomial is $X - 2\alpha$. Then $N^3 = \mathbb{R}((c))$ and $[c, x] = 2\alpha c$. As L/N^3 is as in (3), we can take linearly independent elements a_1, a_2, a_3 for N such that $N = \mathbb{R}((a_1, a_2, a_3)) + N^3$ with products $[a_1, x] \equiv a_2 \pmod{N^3}$, $[a_2, x] \equiv -a_1 + \alpha a_2 \pmod{N^3}$, $[a_3, x] \equiv \alpha a_3 \pmod{N^3}$, $[a_1, a_2] \equiv a_3 \pmod{N^3}$, $[a_1, a_3] \equiv [a_2, a_3] \equiv 0 \pmod{N^3}$. Notice that $N^3 \leqslant Z(N)$. Then by the Jacobi identity we obtain the following:

$$\begin{aligned} [[a_1, a_3]x] - 2\alpha[a_1, a_3] &= [a_2, a_3] - \alpha[a_1, a_3] = 0, \\ [[a_2, a_3]x] - 2\alpha[a_2, a_3] &= -[a_1, a_3] = 0. \end{aligned} \tag{3.3.2}$$

Consequently, $a_3 \in Z(N)$, and this yields that $N^3 = 0$, a contradiction. Therefore the minimal polynomial of $\mathrm{ad}_L\, x|_{N^3}$ is $X^2 - 3\alpha X + (2\alpha^2 + 1)$. Let $\pi(X) = X^2 - \alpha X + 1$ and $\mu(X) = X^2 - 3\alpha X + (2\alpha^2 + 1)$. Notice that in the case $\alpha = 0$, the characteristic polynomial of $\mathrm{ad}_L\, x|_N$ is $(X^2 + 1)^2 X$. We have two possibilities:

Case 1: $\mathrm{ad}_L\, x|_N$ *is semisimple.* Then N is $\mathrm{ad}_L\, x$-completely reducible (see [4, p. 129]). Consequently, we can decompose N as $V \dot{+} N^2$, where V is $\mathrm{ad}_L\, x$-stable. As $N_\alpha \leqslant N^2$, we deduce that the minimal polynomial of $\mathrm{ad}_L\, x|_V$ is $X^2 - \alpha X + 1$. Moreover, N^2 has a decomposition as $N_\alpha \dot{+} N^3$. Notice that $N_\alpha = \mathbb{R}((c))$. Now, take a basis a_1, a_2 for V such that $[a_1, x] = a_2$, $[a_2, x] = -a_1 + \alpha a_2$. Then $N^2 = \mathbb{R}(([a_1, a_2], [a_1, c], [a_2, c]))$. As $\dim N^2 = 3$, we conclude that $[a_1, a_2], [a_1, c], [a_2, c]$ is a basis for N^2. Now, by using the Jacobi identity, we can easily check that $[a_1, a_2] = \lambda c$ with $0 \neq \lambda \in \mathbb{R}$ and therefore $a_1, a_2, \lambda c, \lambda[a_1, c], \lambda[[a_1, c]x], x$ is a basis for L as in (6).
Case 2: $\mathrm{ad}_L\, x|_N$ *is not semisimple.* Then its minimal polynomial has the form $(X^2 + 1)^2 X$. Consequently, we can decompose $N = N_{\pi(X)} \dot{+} N_0$. In that case, $\mathrm{ad}_L\, x$ must act on $N_{\pi(X)}$ cyclically. Take a basis b_1, b_2, b_4, b_5 for $N_{\pi(X)}$ such that $[b_1, x] = b_2$, $[b_2, x] = -b_1 + b_4$, $[b_4, x] = b_5$, $[b_5, x] = -b_4$. Pick $c \in N_0$. Then $N_0 = \mathbb{R}((c))$ and $[c, x] = 0$. Notice that the only $\mathrm{ad}_L\, x$-invariant subspaces of $N_{\pi(X)}$ are $\mathrm{Ker}\, \pi(\mathrm{ad}_L\, x|_N)$ and $N_{\pi(X)}$. Thus $N^3 = \mathrm{Ker}\, \pi(\mathrm{ad}_L\, x|_N) = \mathbb{R}((b_4, b_5))$. As $\dim N^2 = 3$, we conclude that $[b_1, b_2], [b_1, c], [b_2, c]$ is a basis for N^2. Now by the Jacobi identity, it is easily checked that $[b_1, b_2] = \lambda c$, $[b_1, \lambda c] = \beta b_4 + \mu b_5$, and $[b_2, \lambda c] =$

$-\mu b_4 + \beta b_5$, where $0 \neq \lambda$ and $(\beta, \mu) \neq (0,0)$. Write $b_3 = \lambda c$. If $\beta > 0$, consider the following elements:

$$y = K_2 b_3 + x,$$

$$a_1 = K_1 b_1; \quad a_2 = K_1 b_2 + K_1 K_2 \beta b_4 + K_1 K_2 \mu b_5, \quad a_3 = (K_1)^2 b_3,$$

$$a_4 = K_1(1 - 2K_2 \mu) b_4 + 2K_1 K_2 \beta b_5,$$

$$a_5 = -2K_1 K_2 \beta b_4 + K_1(1 - 2K_2 \mu) b_5,$$

where

$$K_1 = \sqrt{\frac{\beta}{\beta^2 + \mu^2}} \quad \text{and} \quad K_2 = \frac{\mu}{2(\beta^2 + \mu^2)}.$$

It is immediate that $\{a_1, a_2, a_3, a_4, a_5, y\}$ is a basis for L as in (7). If $\beta < 0$, consider the following elements:

$$y = K_2 b_3 - x,$$

$$a_1 = K_1 b_1, \quad a_2 = -K_1 b_2 + K_1 K_2 \beta b_4 + K_1 K_2 \mu b_5, \quad a_3 = -(K_1)^2 b_3,$$

$$a_4 = K_1(1 + 2K_2 \mu) b_4 - 2K_1 K_2 \beta b_5,$$

$$a_5 = -2K_1 K_2 \beta b_4 - K_1(1 + 2K_2 \mu) b_5,$$

where

$$K_1 = \sqrt{\frac{-\beta}{\beta^2 + \mu^2}} \quad \text{and} \quad K_2 = \frac{-\mu}{2(\beta^2 + \mu^2)}.$$

It is immediate that $\{a_1, a_2, a_3, a_4, a_5, y\}$ is a basis for L as in (7). If $\beta = 0$, it follows $\mu \neq 0$. Then consider the following basis $\{a_1, a_2, a_3, a_4, a_5, y\}$ for L: if $\mu > 0$.

$$K = \sqrt{\mu} \quad \text{and} \quad y = x,$$

$$a_1 = \frac{1}{K} b_1, \quad a_2 = \frac{1}{K} b_2, \quad a_3 = \frac{1}{K^2} b_3, \quad a_4 = \frac{1}{K} b_4, \quad a_5 = \frac{1}{K} b_5;$$

if, $\mu < 0$,

$$K = \sqrt{-\mu} \quad \text{and} \quad y = \frac{1}{\mu} b_3 + x,$$

$$a_1 = \frac{1}{K} b_1, \; a_2 = \frac{1}{K} b_2 + \frac{1}{K} b_5, \; a_3 = \frac{1}{K^2} b_3, \; a_4 = -\frac{1}{K} b_4, \; a_5 = -\frac{1}{K} b_5.$$

It is easily checked that L is as in (8).

(d) Let L be as in Theorem 2.3 (ix), so that the characteristic polynomial of $\mathrm{ad}_L\, x|_N$ has one of the following forms: $(X - \alpha)(X^2 - c_1 X - c_0)$ or $(X^2 - c_1 X - c_0)(X^2 - d_1 X - d_0)$ with $\alpha \neq 0$, $c_1^2 + 4c_0 < 0$, $d_1^2 + 4d_0 < 0$, and $(c_1, c_0) \neq (d_1, d_0)$. In the first case, taking $y = \lambda x$ with $\lambda = 1/\alpha$, we obtain that the characteristic polynomial of $\mathrm{ad}_L\, y|_N$ is $(X - 1)(X^2 - \mu X - \beta)$ with $\mu^2 + 4\beta < 0$. Now, we can pick a basis a_1, a_2, a_3 for N such that $[a_1, y] = a_1, [a_2, y] = a_3, [a_3, y] = \beta a_2 + \mu a_3$. Consequently L is as in (9). In the second case, we can assume without loss of generality that the minimal polynomial of $\mathrm{ad}_L\, x|_N$ is $(X^2 - \alpha X + 1)(X^2 - \gamma X - \delta)$, where $-2 < \alpha < 2$, $\gamma^2 + 4\delta < 0$, and $(\gamma, \delta) \neq (\alpha, -1)$. Now, we can take a basis a_1, a_2, a_3, a_4 for N such that $[a_1, x] = a_2$, $[a_2, x] = -a_1 + \alpha a_2$, $[a_3, x] = a_4$, $[a_4, x] = \delta a_3 + \gamma a_4$. It is easily checked that two Lie algebras of these families corresponding to scalars (α, β, μ) and (α', β', μ') are isomorphic if and only if $(\alpha', \beta', \mu') = (\mu/\sqrt{-\beta}, 1/\beta, \alpha/\sqrt{-\beta})$, $(-\mu/\sqrt{-\beta}, 1/\beta, -\alpha/\sqrt{-\beta})$, or $(-\alpha, \beta, -\mu)$. From the above isomorphism characterization, the restrictions $0 \leqslant \alpha < 2$ and $-1 \leqslant \beta < 0$ are immediate. Now, if $\beta \neq -1$, $0 \leqslant \alpha < 2$, we conclude that L is as in (10) or (12). Notice that in case (12) we can take $\mu \geqslant 0$. If $\beta = -1$, we obtain that L is as in (11) or (12).

The converse follows from Theorem 2.3. ∎

EXAMPLE 3.4. The following examples ensure the existence of Lie algebras of the type in Theorem 2.3(vi) with nonabelian nilradical. A final remark: if L is as in Theorem 2.3(vi), the structures of $\mathrm{Nil}(L)$ and the Levi factor of L influence each other.

(1) Let N be the nilpotent 3-dimensional Lie algebra with basis $\{a_1, a_2, a_3\}$ having as its only nonzero products $[a_1, a_2] = a_3 = -[a_2, a_1]$. If N is the nilradical of a Lie algebra L as in Theorem 2.3(vi), it is easily checked that L/N must be isomorphic to the split three-dimensional simple Lie algebra. Now, from [5, p. 32], L has a basis $\{a_1, a_2, a_3, x, y, h\}$ with

products $[a_1, a_2] = a_3$, $[a_1, a_3] = 0$, $[a_2, a_3] = 0$, $[x, a_1] = 0$, $[x, a_2] = a_1$, $[x, a_3] = 0$, $[y, a_1] = a_2$, $[y, a_2] = 0$, $[y, a_3] = 0$, $[h, a_1] = a_1$, $[h, a_2] = -a_2$, $[h, a_3] = 0$, $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$. Notice that $\Im(L)$ is the following 4-element chain:

$$0 < F((a_3)) = N^2 < F((a_1, a_2, a_3)) = N < L.$$

(2) Let M be the Lie algebra with basis $\{a_1, a_2, a_3, a_4, a_5, x, y, h\}$ and products $[a_1, a_2] = a_3$, $[a_1, a_3] = a_4$, $[a_2, a_3] = a_5$, $[x, a_1] = 0$, $[x, a_2] = a_1$, $[x, a_3] = 0$, $[x, a_4] = 0$, $[x, a_5] = a_4$, $[y, a_1] = a_2$, $[y, a_2] = 0$, $[y, a_3] = 0$, $[y, a_4] = a_5$, $[y, a_5] = 0$, $[h, a_1] = a_1$, $[h, a_2] = -a_2$, $[h, a_3] = 0$, $[h, a_4] = a_4$, $[h, a_5] = -a_5$, $[h, x] = 2x$, $[x, y] = h$, $[h, y] = -2y$. It is immediate that M is as in Theorem 2.3(vi), where $\Im(L)$ is the following 5-element chain:

$$\begin{aligned} 0 &< F((a_4, a_5)) = \mathrm{Nil}(L)^3 < F((a_3, a_4, a_5)) = \mathrm{Nil}(L)^2 \\ &< F((a_1, a_2, a_3, a_4, a_5)) = \mathrm{Nil}(L) < L. \end{aligned}$$ ∎

REFERENCES

1 M. P. Benito, Lie algebras in which the lattice formed by the ideals is a chain, *Comm. Algebra*, 20(1), 93–108 (1992).
2 P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, 1973.
3 N. Jacobson, *Lie Algebras*, Wiley-Interscience, New York, 1962.
4 N. Jacobson, *Lectures in Abstract Algebras*, Vol. II, Springer-Verlag, 1975.
5 J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.
6 E. J. Marshall, The Frattini subalgebra of a Lie algebra, *J. London Math. Soc.* 42:416–422 (1967).
7 E. L. Stitzinger, On the Frattini subalgebra of a Lie algebra, *J. London Math. Soc.* (2) 2:429–438 (1970).
8 D. A. Towers, A Frattini theory for algebras, *Proc. London Math. Soc.* (3) 27:440–462 (1973).

Received 26 February 1991; final manuscript accepted 21 October 1991