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On Controllability of Timed Continuous Petri Net Systems: the Join Free Case

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***Abstract*— Timed continuous PNs have been used as relaxed models to evaluate 'approximately' the performance of the underlying discrete systems. Moreover, the control of the continuized systems can approximate the scheduling of the (discrete) PNs. This paper analyses the controllability of conservative and consistent join free net systems under infinite server semantics. They are positive systems in which classic control theory is not directly applicable: in this domain input actions are non-negative and dynamically bounded, leading to polytope constrained state space instead of a vectorial space. Thus a new concept of controllability is proposed. The 'controllability space' (CS), included in this polytope, is studied depending on the set of controlled transitions. The full state space is 'controllable' iff all the transitions are controlled. On the other hand, a given state can always be 'controlled' (reached and maintained) without using all transitions. The CS obtained by controlling just one transition is a straight segment, and the CS obtained with several transitions includes the convex of the CS obtained independently with every transition. If additionally the system is choice-free the state space is a partition of the CS obtained with the entire set of transitions except one. Nevertheless borders belong to all neighbour regions.**

I. Introduction

Petri Nets (PNs) constitute a well-known formal paradigm for the modelling, analysis, synthesis and implementation of systems that 'can be seen' as discrete. One of the possible ways to overcome the classical state explosion problem is to continuize the system [1], [2]. Therefore, classical concepts of the control of continuous dynamic systems, like observability [3] and controllability, among others, are analysed here under Infinite Server Semantics (ISS). These continuous PNs are piecewise linear systems whose switches are triggered by internal events. Their special characteristics, mainly the non negativity condition on markings and flows, oblige to apply different control techniques. This fact provokes the control actions to be dynamically bounded by the enabling degree. Additionally, the reachability set in conservative PNs is a *polytope* instead of a *vectorial space*. In this paper the controllability of continuous PNs without synchronizations, i. e., join-free (JF), is studied. They are positive linear dynamic systems [4].

Continuous PNs are presented in section 2. Firstly they are shown under untimed interpretation, afterwards ISS timing interpretation is introduced, and finally an analysis of their properties when they are JF, conservative, and consistent is carried out. In section 3, the bases of the control of PNs with actions associated to the transitions are laid, and their particularities with respect to traditional control theory are explained. Controllability properties of the conservative and consistent JF nets are shown in section 4. Finally, the main conclusions are sketched in section 5.

II. Unforced continuous PNs

A. Untimed systems

Let $N=\langle P,T,\mathbf{Pre},\mathbf{Post}\rangle$ be a PN, where P and T are the places and transitions, and **Pre** and **Post** are matrices that represent the static structure of the net. The state (or fundamental) equation is described by: $\mathbf{m}=\mathbf{m_0}+\mathbf{C}\cdot\sigma$, where $\mathbf{C}=\mathbf{Post}-\mathbf{Pre}$ is the so called incidence (or more properly token flow) matrix, $\sigma\in\mathbb{N}^{|T|}$ is the characteristic vector associated to the firing sequence σ, and $\mathbf{m}, \mathbf{m_0}\in\mathbb{N}^{|P|}$, are the marking and the initial marking, respectively. The reachable set (RS) is composed of all the markings that can be reached from $\mathbf{m_0}$ by means of a firing sequence. Non-expert readers can review, for instance, [5], [6], [1].

Continuous PNs [1], [2], [7] constitute a relaxation of the discrete ones over the non negative real numbers, i. e., with $\mathbf{m}, \mathbf{m_0}\in\mathbb{R}^{+|P|}$, $\sigma\in\mathbb{R}^{+|T|}$ and $\mathbf{m}, \mathbf{m_0}, \sigma \geq 0$. In a continuous net system, each transition, t_i, can be partially fired by α_i, $0\leq\alpha_i\leq\mathbf{e}[t_i]$, where $\mathbf{e}[t_i]$ is the enabling degree of t_i, $\mathbf{e}[t_i]=min_{p\in\bullet t_i}\{\mathbf{m}[p]/\mathbf{Pre}[p, t_i]\}$. The resulting marking after the firing is $\mathbf{m}=\mathbf{m_0}+\alpha_i\cdot\mathbf{C}[P, t_i]$. Given that the continuous PNs that we are considering are consistent JF and every transition is fireable, the set of reachable markings is equal to the solutions of the state equation, $RS=\{\mathbf{m} \mid \mathbf{m_0}+\mathbf{C}\cdot\sigma, \sigma\in\mathbb{R}^{+}\}$ [8].

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B. Timed Systems: Infinite Server Semantics

Deriving with respect to time the fundamental equation of timed continuous PN, $\dot{\mathbf{m}}=\mathbf{C}\cdot\mathbf{f}$ is obtained, where $\mathbf{f}=\dot{\sigma}$ is the flow through the transitions. There are several ways (semantics) to define **f** [1], [9]. ISS (variable speed in [1]) is considered in this paper; under this semantics, the flow of transitions is defined as follows:

$$\mathbf{f}(\tau)[t_i] = \lambda[t_i]\cdot\mathbf{e}(\tau)[t_i]$$
$$\text{with } \mathbf{e}(\tau)[t_i] = \min_{p\in{}^{\bullet}t_i} \{\mathbf{m}(\tau)[p]/\mathbf{Pre}[p,t_i]\} \qquad (1)$$

where $\mathbf{e}(\tau)[t_i]$ is the enabling degree of t_i, which represents the number of active servers in the station (transition) in the instant τ, and $\lambda[t_i]$ is the rate associated to t_i, with $\mathbf{f}(\tau)$, λ, $\mathbf{e}(\tau)\in\mathbb{R}^{+|T|}$. In consequence, the continuized model is a piecewise linear differential equation system [2]. For the sake of convenience, f_i, e_i, λ_i are going to be used to denote $\mathbf{f}(\tau)[t_i]$, $\mathbf{e}(\tau)[t_i]$, $\lambda[t_i]$, respectively. The matrix Λ is defined as $\Lambda = \mathbf{I}\cdot\lambda \in \mathbb{R}^{+|T|} \times \mathbb{R}^{+|T|}$, where **I** is the identity matrix of dimension $|T|$.

C. Conservative and Consistent Timed Join-Free Systems

Let N be a strongly connected, JF, conservative, and consistent net. Conservativeness indicates that there is a positive left annuller of **C** (p-semiflow) that covers all the places ($\exists$ **Y>0** such that $\mathbf{Y}\cdot\mathbf{C}=\mathbf{0}$), what leads to the conservation law ($\mathbf{Y}\cdot\mathbf{m} = \mathbf{Y}\cdot\mathbf{m}_0$). If the net is consistent, there is a positive right annuller of **C** (t-semiflow) that affects all transitions ($\exists$ **X>0** such that $\mathbf{C}\cdot\mathbf{X}=\mathbf{0}$) [6]. JF nets are systems that satisfy $|{}^{\bullet}t_i|=1$ $\forall i\in[0...|T|]$, where ${}^{\bullet}x$ denotes the set of input transitions/places of the place/transition x. If choice-free (CF) PNs ($|p^{\bullet}_i|=1$ $\forall i\in[0...|P|]$) are also considered, some results can be improved ($x^{\bullet}$ denotes the set of output transitions/places of the place/transition x).

Conservative JF nets are mono-p-semiflow, that is to say, they have just one left annuller of **C** linearly independent [10]. By duality, consistent CF nets are mono-t-semiflow. Moreover unforced JF PNs can be converted into equivalent CF-JF nets [11], as it is shown in the example in Figure 1. Strongly connected CF-JF nets have $|P|=|T|$, thus **Pre**, **Post** and **C** are square matrices.

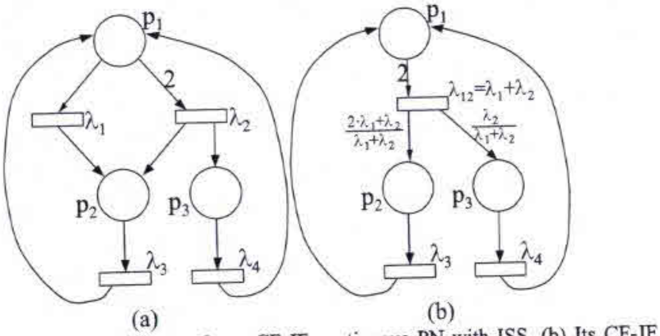

Fig. 1: (a) Example of non-CF-JF continuous PN with ISS. (b) Its CF-JF equivalent in evolution net.

Let us define matrix $\mathbf{W}\in\mathbb{R}^{+|T|}\times\mathbb{R}^{+|P|}$ such that $w_{ij}=1/\text{Pre}[p_j,t_i]$ if an arc from p_i to t_j exists, $w_{ij}=0$ otherwise. Note that if the JF PN is also CF, and places and transitions are appropriately numbered ($p_i={}^{\bullet}t_i$ $\forall p_i\in|P|$), then $\mathbf{W}=\mathbf{Pre}^{-1}$; this will be considered henceforth, without losing generality. Consequently, from (1) the following equation can be deduced:

$$\dot{\mathbf{m}} = \mathbf{C}\cdot\mathbf{f} = \mathbf{C}\cdot\Lambda\cdot\mathbf{e} = \mathbf{C}\cdot\Lambda\cdot\mathbf{W}\cdot\mathbf{m} \qquad (2)$$

Since **C**, Λ and **W** are constant matrices, (2) corresponds to an unforced (without control inputs) linear dynamic system. Now let us show some properties about positiveness and stability.

Proposition 1: Every conservative and consistent JF net is a positive linear system with a (Frobenius) dominant eigenvalue $l_f=0$ that is unique.

Proof: The system defined in (2) is positive because a JF net can be converted into a CF-JF net with a diagonal **Pre** matrix; **Post**, **Pre**≥ 0, and then **C** is a Metzler matrix (its non-diagonal elements are non-negative) and so is **A** ($\mathbf{A}=\mathbf{C}\cdot\Lambda\cdot\mathbf{W}$); this is a sufficient and necessary condition to positiveness in linear systems [4]. On the other hand, the solution of the homogeneous linear equation (2) is $\mathbf{m}(\tau)=e^{\mathbf{A}\tau}\cdot\mathbf{m}_0$. The system is bounded because it is conservative, and then the eigenvalues are non-positive; as **C** is singular so is **A**, and therefore it has a null eigenvalue, which must be dominant. And the dominant eigenvalue of a positive linear system is real and unique by Frobenius' theorem [4]. ■

These systems are known as marginally stable systems or systems with simple stability, in positive linear system theory.

Proposition 2: Conservative and consistent JF nets tend exponentially to its equilibrium marking, $\mathbf{m}_e$, which for an initial marking $\mathbf{m}_0$ is described by the system:

$$\mathbf{C}\cdot\Lambda\cdot\mathbf{W}\cdot\mathbf{m}_e = 0$$
$$\mathbf{Y}\cdot\mathbf{m}_e = \mathbf{Y}\cdot\mathbf{m}_0, \qquad (3)$$

where **Y** is the conservative component of the net.

Proof: $\mathbf{m}_e$ must satisfy both equations because it is an equilibrium point and verifies the conservation law. Since a JF net can be reduced into a CF-JF net, that is, a mono-t-semiflow net, then the markings satisfying $\mathbf{C}\cdot\Lambda\cdot\mathbf{W}\cdot\mathbf{m}_e=0$ represent a one dimensional space. The intersection of that space with $\mathbf{Y}\cdot\mathbf{m}_e=\mathbf{Y}\cdot\mathbf{m}_0$ yields the only marking the system can evolve to.■

It is important to note that if the conservative net is not JF, then a bounded input-bounded output system can oscillate without damping around the equilibrium marking, $\mathbf{m}_e$, which is never reached. An example is shown in [12].

Let us consider the system in Figure 1(*a*). It is a homogeneous linear system $\dot{\mathbf{m}}=\mathbf{C}\cdot\Lambda\cdot\mathbf{W}\cdot\mathbf{m}$, described by:

$$\mathbf{C}\cdot\Lambda\cdot\mathbf{W} = \begin{bmatrix} -\lambda_1-\lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1-\lambda_2/2 & -\lambda_3 & 0 \\ \lambda_2/2 & 0 & -\lambda_4 \end{bmatrix}$$

The three eigenvalues of the system are:

$$\text{eig} = [0,\ -\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2}{2} \pm\sqrt{(\lambda_1+\lambda_2)^2+(\lambda_3-\lambda_4)^2+2\cdot\lambda_1(\lambda_3+\lambda_4)}]$$

This system has a p-semiflow ([1 1 1]) and two t-semiflows ([1 0 1 0], [0 1 1 1]). Its evolution tends exponentially to the equilibrium marking. For an initial marking that verifies $\mathbf{Y}\cdot\mathbf{m}_0=\mathbf{1}$, that position is $\mathbf{m}_e=(2\cdot\lambda_3\cdot\lambda_4+2\cdot\lambda_1\cdot\lambda_4+\lambda_2\cdot\lambda_3+\lambda_2\cdot\lambda_4)^{-1}\cdot[2\cdot\lambda_3\cdot\lambda_4;2\cdot\lambda_1\cdot\lambda_4+\lambda_2\cdot\lambda_4;\lambda_2\cdot\lambda_3]$. Figure 2 shows the evolution of the system when $\boldsymbol{\lambda}=\mathbf{1}$ and $\mathbf{m}_0=[1\ 0\ 0]$, first with respect to time, and afterwards in the state space for several initial markings. Note that *the state* is determined with the marking of only two places, due to the conservative component, which provides the marking of the third place.

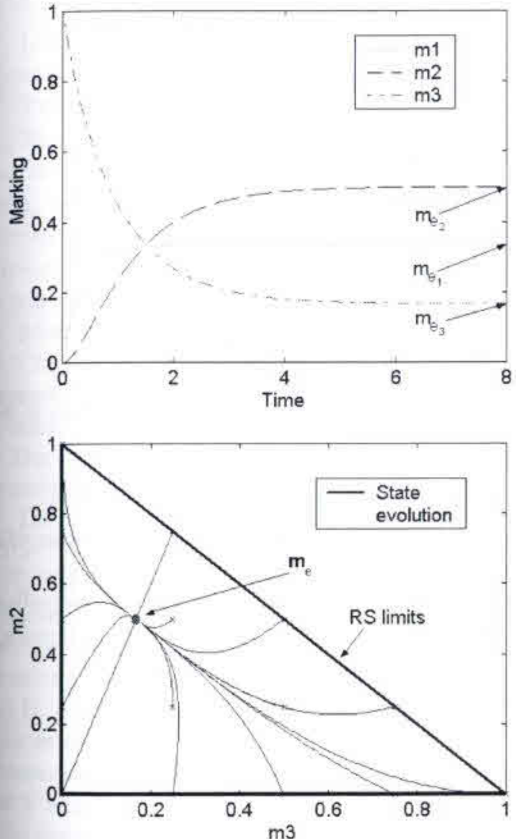

Fig. 2: Evolution of the marking of the system in fig. 1*a* when $\boldsymbol{\lambda}=\mathbf{1}$. (a) Temporal evolution for $\mathbf{m}_0=[1\ 0\ 0]$; (b) Evolution in the state space for several initial markings.

III. Forced Continuous Petri Nets

A. Controlling the Flow of Transitions

Let us suppose that it is possible to apply certain control action to the transitions that permits to decrease their flow. This 'slowing down' action can be included in the flow equation with an additional term, **u**.

Definition 1: A *control action* $u(\tau)[t_i]$ of a transition t_i is a time-dependent variable that represents the reduction of the enabling in the flow of the transition. The transition in which there exists the possibility to apply a control action is called *controlled transition*, and its flow is $f(\tau)[t_i]=\lambda[t_i]\cdot(e(\tau)[t_i]-u(\tau)[t_i])$. Using again u_i instead of $u(\tau)[t_i]$, for the sake of convenience, with $W_i=W[t_i,{}^\bullet t_i]$:

$$\begin{aligned} f_i &= \lambda_i\cdot(e_i-u_i)=\lambda_i\cdot(W_i\cdot m_i-u_i)\\ 0 &\le u_i\le e_i=W_i\cdot m_i \end{aligned} \qquad (4)$$

Note that actions u_i are defined as a 'reduction' of the enabling degree. Then $u_i\geq 0$ and they have to be smaller than or equal to the enabling so that flows are non negative, $u_i\leq e_i$.

A *control vector* $\mathbf{u}\in\mathbb{R}^{|T|}$ can be defined, such that each component u_i represents the action on the transition t_i (therefore $u_i=0$ for any uncontrolled transition). Accordingly:

$$\begin{aligned} \dot{\mathbf{m}} &= \mathbf{C}\cdot\boldsymbol{\Lambda}\cdot(\mathbf{W}\cdot\mathbf{m}-\mathbf{u})\\ \mathbf{0} &\le \mathbf{u}\le\mathbf{e}=\mathbf{W}\cdot\mathbf{m} \end{aligned} \qquad (5)$$

Let us define the residual enabling as the enabling after decreasing the control action, i.e., $\mathbf{e}_R=\mathbf{W}\cdot\mathbf{m}-\mathbf{u}$. Thus (5) is rewritten as:

$$\begin{aligned} \dot{\mathbf{m}} &= \mathbf{C}\cdot\boldsymbol{\Lambda}\cdot\mathbf{e}_r\\ \mathbf{0} &\le \mathbf{e}_r\le\mathbf{W}\cdot\mathbf{m} \end{aligned} \qquad (6)$$

The equation (5) represents a linear time-invariant system, (Figure 3). But it must be considered that $\mathbf{0}\leq\mathbf{u}\leq\mathbf{e}$, i.e., $\mathbf{e}_r(\tau)$ cannot be negative, as it is shown in discontinuous line in Figure 3. For that reason, controllability in continuous PN, which will be considered in the next section, differs from its presentation in classical control theory of linear systems.

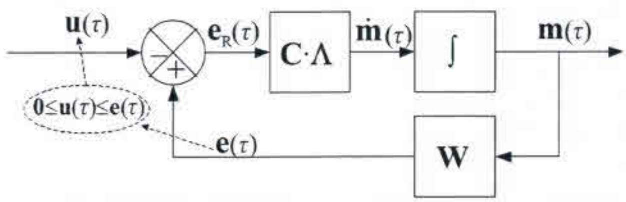

Fig. 3: Block diagram representation of JF continuous PNs, as linear time-invariant systems.

Note that according to (5) the equations of a JF net with control actions on the transitions, when constraints on the actions are not considered, do not 'strictly' correspond to a positive system: the *input matrix* $\boldsymbol{B}=-\mathbf{C}\cdot\boldsymbol{\Lambda}$ does not satisfy the non-negativity condition [4]. Nevertheless, the marking of the net is always non-negative, due to the upper bound constraints on the control action **u**.

In general, forced JF nets cannot be converted into CF-JF nets, as seen with unforced nets. For instance, when the transitions of the system of Fig. 1a are controlled, the weights in the arcs from t_{12} to p_2 and p_3 in Figure 1(*b*) must be respectively

$$\frac{-m_1\cdot\lambda_2/2+\lambda_2\cdot u_2}{-m_1(\lambda_1+\lambda_2)+\lambda_1\cdot u_1+2\cdot\lambda_2\cdot u_2} \qquad \frac{-m_1(\lambda_1+\lambda_2/2)+\lambda_1\cdot u_1+\lambda_2\cdot u_2}{-m_1(\lambda_1+\lambda_2)+\lambda_1\cdot u_1+2\cdot\lambda_2\cdot u_2}$$

in order to get equivalence with the system in Figure 1(*a*). These weights depend on the marking; then a structural transformation cannot be made from one net to the other one.

B. *On Controllability in Petri Nets with Control Actions on the Transitions*

Classical control theory defines that a system is controllable in an instant t_0 if it is possible to drive the state from an initial value $\mathbf{x}(\tau_0)$ to any other state by means of an unconstrained control vector, in a finite time interval [13], [14].

That classic definition cannot be directly applied to conservative continuous PNs because of two facts:

– The dynamic constraints of the actions required for the positiveness of the markings, flows, and actions.

– The reachable marking limitation to a polytope (in contrast to the vectorial space analysis, unsatisfactory for PN based works [15]).

This limitation to the reachablility space is caused by the existence of the conservation laws ($\exists$ $\mathbf{Y}>\mathbf{0}$ $\mathbf{Y}\cdot\mathbf{C}=\mathbf{0}$), which means that $\mathbf{A}$ is not a full rank matrix.

In classical system theory a 'controllable' state (reachable and maintainable) can be reached in finite time. However in PNs, due to the constraints on the actions, the time required to reach some markings (if emptying places) may be infinite.

Thus, the classical definitions have to be adapted as follows:

Definition 2: A marking $\mathbf{m}_f$ is said to be *controllable* from an initial state $\mathbf{m}_0$ by controlling transitions $Tc\subseteq T$ when there exists a control action $\mathbf{u}(\tau)$ on Tc able to drive the marking from $\mathbf{m}_0$ to $\mathbf{m}$ ($\mathbf{m}_0 \xrightarrow{\mathbf{u}} \mathbf{m}_f$), in finite or infinite time, and maintains this marking (if $\mathbf{m}=\mathbf{m}_f$, then $\dot{\mathbf{m}}(\mathbf{u})=0$).

Definition 3: A state $\mathbf{m}_f$ is said to be *temporarily controllable* from an initial state $\mathbf{m}_0$ with control on a set of transitions $Tc\subseteq T$ iff by means of control actions $\mathbf{u}$ on Tc the marking $\mathbf{m}_f$ can be reached from any $\mathbf{m}_0\in RS$.

From the above, the following spaces can be defined:

Definition 4: Given an initial marking $\mathbf{m}_0$ and a set of controlled transitions $Tc\subseteq T$, the *Controllability Space* (CS) is defined as the set of all the controllable markings, i.e., $CS=\{\mathbf{m}_f \mid \exists\ \mathbf{u}(\tau)$ such that $\mathbf{m}_0 \xrightarrow{\mathbf{u}} \mathbf{m}_f$; when $\mathbf{m}=\mathbf{m}_f$ then $\dot{\mathbf{m}}(\mathbf{u})=0\}$. In the same way, the *Temporarily Controllability Space* (TCS) is the set of the temporarily controllable markings.

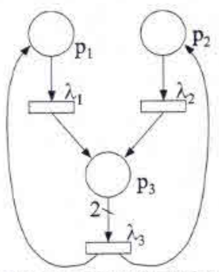

Fig. 4: Example of a CF-JF net interpreted with control on its transitions.

From previous definitions, it can be deduced that CS is an 'attractor' of every marking, i.e., from every state the system evolves towards CS sooner or later. TCS is also an attractor (since $CS\subseteq TCS$) but additionally once the state has reached the TSC, it can never be taken out of it.

Let us consider for instance the PN in Figure 4, with $\mathbf{Y}\cdot\mathbf{m}_0=\mathbf{1}$ and $\lambda=\mathbf{1}$, whose CS and TCS are shown in Figure 5. Each space is obtained as the *convex* defined by the lines shown in Table 1, as a function of the controlled transitions $t_i\in Tc$.

TABLE 1
CONTROLLABILITY SPACES AND TEMPORARILY CONTROLLABILITY SPACES IN THE CF-JF NET SYSTEM SHOWN IN FIGURE 4 (AND 5).

	t1	t2	t3	t1,t2	t2,t3	t3,t1	t1,t2,t3
CS	4	5	6	3-4-5	1-5-6	2-4-6	1-2-3
TCS	9-10	7-8	6	3-8-10	1-6-7	2-6-9	1-2-3

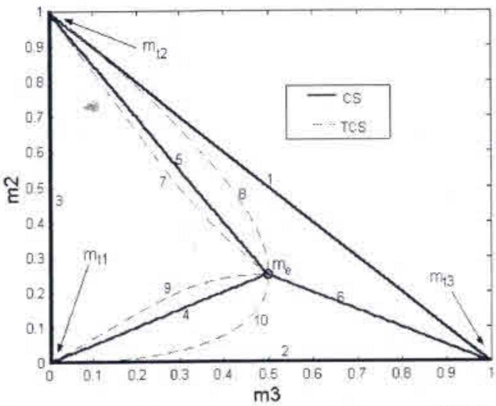

Fig. 5: Controllability spaces and temporarily controllability spaces in the PN shown in Figure 4

IV. CONTROLLABILITY OF JOIN-FREE PETRI NETS

This section shows the main controllability properties of a connected conservative and consistent JF continuous net systems, with initial marking $\mathbf{m}_0$, rates λ_i and controlled transitions $Tc\subseteq T$, which can be controlled with actions u_i. Let us remember that $0\leq u_i\leq e_i$ and $u_i=0$ $\forall t_i\notin Tc$.

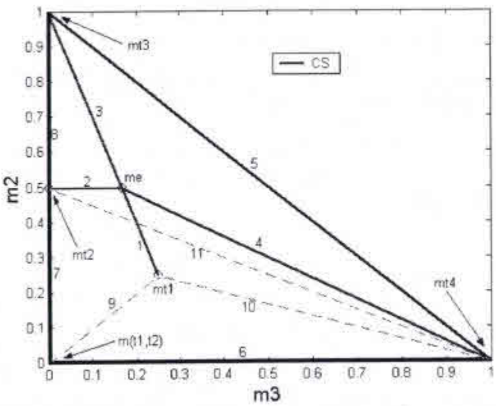

Figure 6: Controllability Spaces of the non-CF-JF net system in Figure 1*a*. The CS are the convex spaces generated by m_e and m_{ti} $\forall t_i\in Tc$, and if $t_1,t_2\in Tc$ then $m(t_1\text{-}t_2)$ must be also included.

The properties have been divided into three types, regarding controllability spaces, controlled transitions, and control actions. Most of them can be deduced from (5), bearing in mind that a marking is controllable when $\dot{\mathbf{m}}=0$, so $\Lambda\cdot(\mathbf{W}\cdot\mathbf{m}-\mathbf{u})$ must be a non-negative right annuller of $\mathbf{C}$. Consequently:

$$\mathbf{X}=\Lambda\cdot(\mathbf{W}\cdot\mathbf{m}-\mathbf{u}) \Rightarrow$$

$$\mathbf{u}=\mathbf{W}\cdot\mathbf{m}-\Lambda^{-1}\cdot\mathbf{X} \quad (7)$$

$$\mathbf{W}\cdot\mathbf{m}=\Lambda^{-1}\cdot\mathbf{X}+\mathbf{u} \quad (8)$$

The CF-JF net in Figure 4, whose control zones are represented in Figure 5, and the non-CF-JF net system in Figure 1*(a)*, whose CS regions are represented in Figure 6, will be used to exemplify the properties.

A. On the Controllability Spaces

Let us analyse properties of the different CS obtained by controlling transitions.

Proposition 3: CS is a convex space.

Proof: Let $\mathbf{m}_1$, $\mathbf{m}_2 \in$ CS and $\mathbf{u}_1$, $\mathbf{u}_2$ their associated control actions. Due to the linearity of (8) the control action $\alpha\cdot\mathbf{u}_1+(1-\alpha)\cdot\mathbf{u}_2$ with $0\le\alpha\le1$ fulfils the constraints in (5), and drives the system to marking $\mathbf{m}(\alpha\cdot u_1+(1-\alpha)\cdot u_2)=\mathbf{m}_1\cdot\alpha+\mathbf{m}_2\cdot(1-\alpha)$. Thus any point between $\mathbf{m}_1$ and $\mathbf{m}_2$ belongs to CS.■

Moreover:

- The CS corresponding to a PN system in which every transition is controlled, i. e., Tc=T, is a polytope that coincides with the RS of the untimed continuous PN.
- Due to the adopted ISS, it can be observed that it is not possible to drive the marking to the CS border in finite time.

Let us consider three different system behaviours when |Tc|=1. Let us assume that t_i is the only controlled transition and that a maximum control action is applied to it, i. e., t_i is blocked ($\mathbf{u}_i=\mathbf{e}_i$). Let us also denote $p_i={}^{\bullet}t_i$. The net system always evolves to one of the following types of stable markings, depending on the structure of the net:

a) If $|p_i^{\bullet}|=1$, then the system evolves to a state with all the places empty but p_i, whose marking is $\mathbf{Y}\cdot\mathbf{m}_0/Y_i$, where $\mathbf{Y}$ is a conservative component of the PN and Y_i is the component of $\mathbf{Y}$ which corresponds to p_i. That state is a vertex of the CS (for instance m_{t3} and m_{t4} in Figure 6).

b) If $|p_i^{\bullet}|>1$, but $\exists\, p_j\in t_i^{\bullet}$ such that $|{}^{\bullet}p_j|=1$, then the system evolves to a marking that is null at least in the places p_j, which corresponds to a marking in the CS border (m_{t2} in Figure 6).

c) If $|p_i^{\bullet}|>1$ and $\forall p_j\in t_i^{\bullet}$ $|{}^{\bullet}p_j|>1$, then the system evolves to a marking inside the CS, with no empty places (m_{t1} in Fig. 6).

If the system is CF, case *(a)* is the only possible one (Fig. 5).

In any of the previous cases the reached marking is known as *marking with blocked* t_i and it is denoted as $\mathbf{m}_{ti}$.

Given (8) and provided that CS is a convex space, the set of controllable markings is just a segment if only one transition is controlled.

Proposition 4: If Tc={t_i}, the CS of the system is a (rectilinear) segment between the equilibrium marking without actions, $\mathbf{m}_e$, and the marking obtained with blocked t_i, $\mathbf{m}_{ti}$.

Proposition 5: If various transitions are controlled, the CS of the system is a polytope that includes the convex generated by the CS corresponding to each transition. If the system is also CF, the region is exactly equal to the convex.

Proof: The first statement of the proposition follows directly from the fact that CS is convex. Additionally, in CF nets, the control actions applied to the transitions satisfy $\Sigma k_i\cdot u_i\le\Sigma k_i\cdot u_{i,max}$ $\forall i$ such that $t_i\in$Tc, where $u_{i,max}$ represents the maximum control action that can be applied to transition i for any marking in CS. Thus no marking out of the convex can be controllable. ■

For instance, the CS with t_1 and t_2 in Figure 6 (region 1-2-7-9) is bigger than the convex of the CS corresponding to t_1 and t_2 independently ($\mathbf{m}_e$-$\mathbf{m}_{t1}$-$\mathbf{m}_{t2}$).

Considering proposition 4 and 5 for CF nets, and bearing in mind that in those cases m_{ti} are vertices of the CS, the following result is obtained:

Proposition 6: If the system is CF and Tc=T, then the CS of the system is a polytope defined by the union of all the CS formed by controlling all the transitions with the exception of one. Moreover the intersections between neighbour sets are only the borders. That is to say, if CS_T is the CS for T=Tc and CS_{Ti} is the CS for T_i=Tc−{t_i}, then $\cup_i CS_{Ti}=CS_T$ and $CS_{Ti}\cap CS_{Tj}=F_{ij}$ $\forall i\neq j$, where the geometric dimension of the border F_{ij} (between RC_{Ti} and RC_{Tj}) is $\dim(F_{ij})=\dim(T_i)-1$.

This proposition can be exemplified in Figure 5, where the whole CS, region 1-2-3, can be divided into the three regions 1-5-6, 2-6-4, 3-4-5, which constitute a partition.

From previous definitions it can be observed that if a set of transitions Tc is controlled, the CS is a polytope with dimension $d\le min\{|Tc|, |P|-1\}$. If the system is also CF then $d=min\{|Tc|, |P|-1\}$.

B. On the Controlled Transitions

The basic objective in this subsection is to determine which is the set of transitions that are needed in order to control (to reach and maintain) a certain state of the system.

Proposition 7: The only way to control a pure JF net system on the whole RS is to control all the transitions. However, given any particular (reachable) marking $\mathbf{m}_f$, it is always controllable with a set of controlled transitions Tc such that |Tc|<|T|.

Proof: If a place has several output transitions, all of them must be controlled 'to fill' completely the place. Therefore, all transitions are needed. On the other hand, from (7) vector $\mathbf{X}$ can be defined large enough that at least one component of $\mathbf{u}$ decreases just to zero. These components correspond to the transitions that do not need to be controlled. ■

Considering the directions from the non-forced equilibrium marking, $\mathbf{m}_e$, to the markings obtained by blocking every transition, $\mathbf{m}_{ti}$, it can be observed that a necessary condition so that a marking $\mathbf{m}_f$ can be maintained with a subset of controlled transitions Tc is the existence of constants $k_i\ge0$ such that

$$\mathbf{m}_f - \mathbf{m}_e = \sum_{i=1}^{|Tc|} k_i \cdot (\mathbf{m}_{fi} - \mathbf{m}_e)$$

From previous propositions, it can be deduced that the minimum set of transitions required to control a marking do not need to be unique. Nevertheless, if the system is CF the solution is unique.

Proposition 8: Given a marking $\mathbf{m}$ and a subset of controlled transitions $Tc \subseteq T$, $\mathbf{m}$ is controllable iff it belongs to the convex formed by the equilibrium marking, $\mathbf{m}_e$, and *the markings with t_c blocked*, $\mathbf{m}_{fi}$, for every subset of transitions $tq \subseteq Tc$, $\mathbf{m}_{tq}$.

If the net is CF, given a state $\mathbf{m}$ and a subset of controlled transitions Tc, the system can be maintained at $\mathbf{m}$ iff it belongs to the convex formed by the CS of each $t_i \in Tc$.

Proof: From the linearity of markings and actions, it can be deduced that the border of the CS would only be reached when the applied actions are maximum. Then, when all the combinations of the maximum actions are applied to the controlled transitions, all the vertices of the polytope CS are covered, and its convex is the complete CS. ■

C. *On the Control Actions.*

The main goal in this subsection is to know the control actions that must be applied to control (to reach and maintain) a certain state.

Proposition 9: Given a constant control action $\mathbf{u}_f$ there is a unique controllable marking associated to it. Nevertheless, given a marking $\mathbf{m}_f > \mathbf{0}$, there are several control actions that can maintain it.

Proof: From (8) and due to the conservation law, it is immediate to observe that there is a unique equilibrium marking. Furthermore, the second statement is deduced from (8) and bearing in mind that **X** can be formed by any linear combination of its components. ■

Additionally, if the action $u_i(\tau) = min\{u_{fi}, e_i(\tau)\}$ is applied to every transition, then the systems evolves to $\mathbf{m}_f$. Since ISS is being used, this evolution is asymptotic.

For instance, in the PN of Figure 1(*a*), the marking to which the system will evolve is, depending on **u**:

$$m_{f1} = 2 \cdot k_1 \cdot \lambda_3 \cdot \lambda_4 + 2 \cdot u_2$$
$$m_{f2} = 2 \cdot k_1 \cdot \lambda_1 \cdot \lambda_4 + k_1 \cdot \lambda_2 \cdot \lambda_4 - (\lambda_1/\lambda_3) \cdot u_1 + (2 \cdot \lambda_1/\lambda_3) \cdot u_2 + u_3$$
$$m_{f3} = k_1 \cdot \lambda_2 \cdot \lambda_3 + u_4$$

with $k_1 = (m_o + (\lambda_1/\lambda_3) \cdot u_1 - (2+2 \cdot \lambda_1/\lambda_3) \cdot u_2 - u_3 - u_4) / (2 \cdot \lambda_3 \cdot \lambda_4 + 2 \cdot \lambda_1 \cdot \lambda_4 + \lambda_2 \cdot \lambda_3 + \lambda_2 \cdot \lambda_4)$

And the actions **u** are shown below as well as the constraints existing for the parameters k_0 and k_1.

$u_1 = m_{f1} - k_0$	$0 \le k_0 \le m_{f1}/\lambda_3$
$u_2 = m_{f1}/2 - k_1$	$0 \le k_1 \le m_{f1}/(2 \cdot \lambda_3 \cdot \lambda_4)$
$u_3 = m_{f2} - k_0 - k_1$	$k_0 \cdot \lambda_1 + k_1 \cdot \lambda_2 \cdot \lambda_4 \le m_{f2}$
$u_4 = m_{f3} - k_1$	$k_1 \le m_{f3}/\lambda_2 \cdot \lambda_3$

V. Concluding Remarks

Continuous PNs are the result of dropping the integrality constraint in the firing of transitions in 'classical' discrete PNs. This relaxation aims to avoid the state explosion problem and to offer a more satisfactory treatment of highly populated systems. This paper concentrates on controllability in the framework of continuous JF nets under infinite server semantics. Results from system theory cannot be directly applied here for two reasons: 1) The state space is constrained to a polytope 2) The control actions are dynamically bounded.

The CS is a convex space that in the case of a system with just one controlled transition is a straight segment. In the case of a system with several controlled transitions, the CS includes the convex of the CS obtained independently with every transition. Moreover, if the system is choice-free the state space is a partition of the CS obtained with the entire transitions except one. It has been shown that the full state space of the net system is 'controllable' iff all the transitions are controlled. Furthermore, any state is controllable without using all transitions.

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