2007 Mediterranean Conference on Control and
Automation, July 27 - 29, 2007, Athens - Greece

Analytical formulation to compute QFT templates for plants with a high number of uncertain parameters

Juan José Martín-Romero
Department of Electrical Engineering
University of La Rioja, Spain
juanjo@cossio.net

Montserrat Gil-Martínez
Department of Electrical Engineering
University of La Rioja, Spain
montse.gil@unirioja.es

Mario García-Sanz
Department of Automatic Control and
Computer Science
Public University of Navarra, Spain
mgsanz@unavarra.es

Abstract— **This paper introduces an analytical formulation to describe QFT templates based on Fourier series. This allows to calculate the template contour of plants with a high number of uncertain parameters. The proposed solution consists of performing operations between analytical sub-templates. This saves calculation time and allows to work with an infinite set of plant values.**

I. Introduction

Quantitative Feedback Theory enables the design of robust feedback controllers. It was initially proposed by Horowitz [1]–[3]. A survey of the methodology can be found in [4] and full details in [5]–[7]. In a classical two-degree of freedom feedback control structure (see Fig. 1), certain robust performance and stability specifications can be achieved by defining a controller $G(s)$ and a prefilter $F(s)$ according to the QFT methodology.

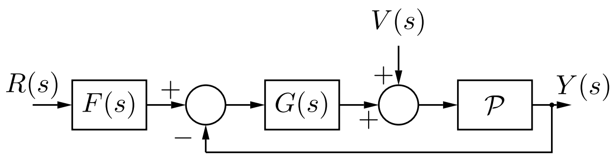

Fig. 1. Two-degree of freedom feedback control structure

In real control systems, the plant model $\mathcal{P}$ contains uncertainty:

$$\mathcal{P} = \frac{N(s, \lambda_1, \lambda_2, \ldots, \lambda_m)}{D(s, \lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n)} \qquad (1)$$

where N and D are the numerator and denominator polynomials. They are functions of the complex variable s and the uncertain parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ where $\lambda_i \in [\lambda_{i_{min}}, \lambda_{i_{max}}]$. For the method proposed in this paper, each uncertain parameter must only appear once in the plant transfer function $\mathcal{P}$. This uncertainty is represented as a value set at each frequency of interest, ω_i, and is known as the ω_i-template. The ω_i-template is depicted on the Nichols Chart (NC) as a bounded surface. See the example[1] shown in Fig. 2.a)

[1] $P(s) = \frac{10\, e^{T_1 s}}{T_2 s+1}$ with $T_1 \in [0.3, 0.9]$ and $T_2 \in [0.5, 1.2]$ at $\omega = 3$ rad/s was used in the example.

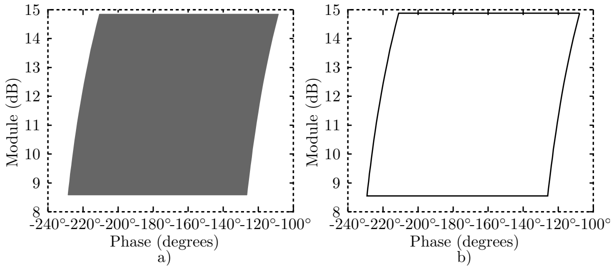

Fig. 2. a) Set of possible values; b) Necessary value set

The templates are used to calculate the QFT bounds that express the closed loop control specifications (robust stability and robust performance) in terms of the open loop nominal system. Afterwards, the controller is designed using the loop-shaping technique to meet the bounds [5]–[7].

Only the edge points of this ω_i-template are significant for bound computation and controller design, as shown in Fig. 2.b). This reduces the computational effort in bound calculation. In this paper the contour of the plant template at ω_i is labelled $\mathcal{T}_{\omega_i}$.

Determining the contour of the value set when the plant has one or two uncertain parameters is usually a simple procedure. In many cases, when the Edge Theorem holds [8], the contour in the parameter space yields the contour in its frequency representation. But, that is not always the case. See for example [9]; for the plant $\mathcal{P} = \frac{e^{-Ts}}{s^2+0.04s+\omega_n^2}$ where $T \in [0, 2]$ and $\omega_n \in [0.7, 1.2]$, at $s = j\omega = j\,1$. Figure 3 compares the template obtained by a grid partition of the contour in the parameter space a) to the true template contour b). Then, the algorithm for template contour computation must be carefully selected, even for simple plants with just two uncertain parameters. Some algorithms [9]–[13] can help to solve this.

When the number of uncertain parameters in the plant is higher than two, the problem becomes more complex and hinders the selection of the set of parameter values that gives the contour points of the ω_i-template, $\mathcal{T}_{\omega_i}$. This study focuses on plants with more than two uncertain parameters that produce simply connected templates.

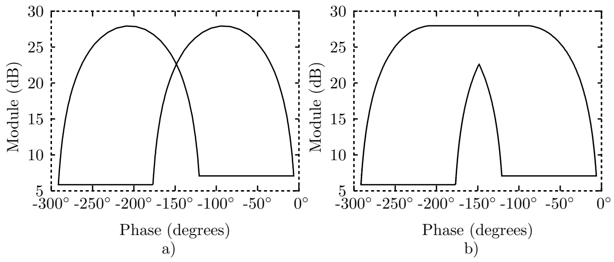

Fig. 3. a) Wrong template contour, b) Correct template

Previous studies have addressed the problem using grid methods. Firstly, the uncertain surface is obtained in the Mod-Arg plane (or in the Imag-Real plane) as shown in Fig. 2.a) by gridding the parameter space. The next step is to find the edge points. This problem is difficult to solve accurately. Besides, it usually requires a huge computational effort. Some optimized grid methods are explained in [12]. For grid methods and non-convex templates, it is possible to select those edge points that are significant in the controller design according to the control specifications, as shown in [14].

This paper proposes an alternative to avoid the shortcomings of grid methods. It consists of performing operations between sub-templates to obtain the final template boundary. This idea, based on a tree decomposition of the plant, has been previously used in other studies. In [15], it is used to build an algorithm for addition and multiplication of templates. The work in [16] also applies this method, although the results are only valid for certain types of plants. Tree decomposition is a relevant factor, even when grid methods are used, as shown in [10].

A survey of template generation methods can be found in [11], which reviews the techniques used until 1996. The problem has still not been resolved and has been addressed using other methods like symbolic computation [9], [17] or interval analysis [18], [19].

This paper follows the initial idea of [15]. The new work presents a method to obtain analytical templates and to define the addition and multiplication between them. The paper is structured as follows. Section II presents a mathematical formulation of the new approach. Some remarks on its advantages and certain special cases are introduced in Section III. Section IV illustrates the new formulation with an example. Finally, the conclusions are presented in Section V.

II. Proposed method

Given a generic plant $\mathcal{P}$ with n uncertain parameters: $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ where $\lambda_i \in [\lambda_{i_{min}}, \lambda_{i_{max}}]$, and using the symbol '$\circ$' to describe a generic arithmetic operation (i.e. $+$, $-$, $\times$, $/$); it is possible to express the plant by means of operations between sub-plants of one and two uncertain parameters:

$$\mathcal{P}(j\omega, \lambda_1, \lambda_2, \ldots, \lambda_n) = \tag{2}$$
$$P_1(j\omega, \lambda_1, \lambda_2) \circ \ldots \circ P_{\frac{k+1}{2}}(j\omega, \lambda_k, \lambda_{k+1}) \circ$$
$$\circ P_{\frac{k+3}{2}}(j\omega, \lambda_{k+2}) \circ \ldots \circ P_{n-1-\frac{k}{2}}(j\omega, \lambda_n)$$

At $\omega = \omega_i$ every sub-plant defines a sub-template $\mathcal{T}^r_{\omega_i}$, i.e. $P_1(j\omega_i, \lambda_1, \lambda_2) \Rightarrow \mathcal{T}^1_{\omega_i}$ or $P_{n-1-\frac{k}{2}}(j\omega, \lambda_n) \Rightarrow \mathcal{T}^{n-1-\frac{k}{2}}_{\omega_i}$, such that the complete template $\mathcal{T}_{\omega_i}$ is,

$$\mathcal{T}_{\omega_i} = \mathcal{T}^1_{\omega_i} \circledcirc \mathcal{T}^2_{\omega_i} \circledcirc \ldots \tag{3}$$

where the symbol $\circledcirc$ is the generic arithmetic operation between templates (it can be addition $\oplus$, subtraction $\ominus$, multiplication $\otimes$, or division $\oslash$). Note that the symbol to operate with templates is different from the symbol to operate with plants because of the different nature of the operations. The templates are actually curves; when a template is operated with another one, every point in the first one operates the complete curve of the second one. Then a family of curves appears. The final result is the envelope curve [20] of this family. Figure 4 shows the family of curves (dashed lines) of the operation $\mathcal{T}^1_{\omega_i}/\mathcal{T}^2_{\omega_i}$ between the templates shown in the example at the end of the paper. It also shows the template contour (solid line) achieved when the operation[2] is $\mathcal{T}^1_{\omega_i} \oslash \mathcal{T}^2_{\omega_i}$.

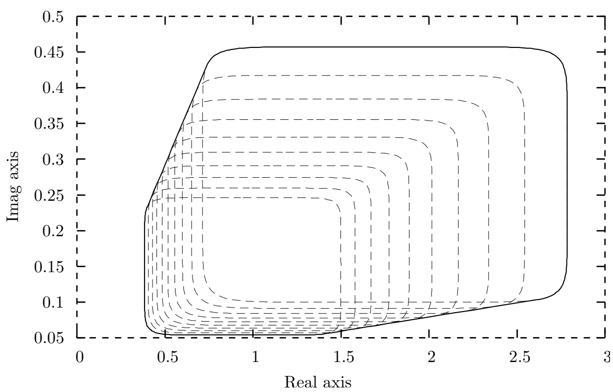

Fig. 4. Family of curves and its envelope curve (the template)

This section is divided in four parts: definition of an analytical template, addition of analytical templates, multiplication of analytical templates and the algorithm proposed.

A. Definition of an analytical template

A template $\mathcal{T}_{\omega_i}$ is a set of infinite complex points that can be represented on Imag-Real or Mod-Arg planes. The classical methodology takes a finite amount n of these points to build the discrete template ${}^d\mathcal{T}_{\omega_i}$:

$$\begin{aligned} {}^d\mathcal{T}_{\omega_i} &= \{p_{r1} + j\,p_{i1}, p_{r2} + j\,p_{i2}, \ldots, p_{rn} + j\,p_{in}\} \\ &= \{p_{m1}\angle p_{a1}, p_{m2}\angle p_{a2}, \ldots, p_{mn}\angle p_{an}, \} \end{aligned} \tag{4}$$

where p_{rk} and p_{ik} are the real and imaginary parts of the k-th point and p_{mk} and p_{ak} are the module and argument of the k-th point. For the algorithm presented in this paper,

[2]Note $\mathcal{T}^1_{\omega_i}/\mathcal{T}^2_{\omega_i}$ is a family of curves and $\mathcal{T}^1_{\omega_i} \oslash \mathcal{T}^2_{\omega_i}$ is the envelope one.

the points in ${}^d\mathcal{T}_{\omega_i}$ must be sorted as they consecutively appear on the contour of the template.

The real-imag array (points $p_{rk} + j\,p_{ik}$) can be used to calculate the analytical template. Then, the arrays ${}^d\mathcal{R} = \{p_{r1}, p_{r2}, \ldots, p_{rn}, p_{r1}\}$ and ${}^d\mathcal{I} = \{p_{i1}, p_{i2}, \ldots, p_{in}, p_{i1}\}$ can be associated with the array ${}^d\varphi = \{0, \frac{2\pi}{n-1}, 2\frac{2\pi}{n-1}, \ldots, (n-2)\frac{2\pi}{n-1}, 2\pi\}$. For example, the discrete functions in Fig. 5 are obtained when the discrete template contour complex points of Fig. 2.b) are taken in rectangular form.

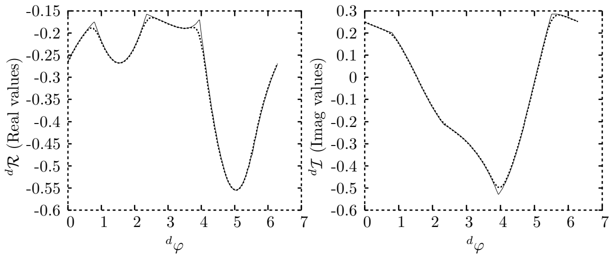

Fig. 5. Imaginary and real curve of the ${}^d\mathcal{T}_{\omega_i}$

Figure 5 also includes the curves with round corners (dashed lines); these new continuous and differentiable curves simplify the fitting of analytical functions. It is possible to approach the discrete curves using splines or polynomials, but the Fourier series are the best solution offering continuous and differentiable functions where $\mathcal{T}_{\omega_i}(0) = \mathcal{T}_{\omega_i}(2\pi)$ and $\left.\frac{d\,\mathcal{T}_{\omega_i}(\varphi)}{d\varphi}\right|_{\varphi=0} = \left.\frac{d\,\mathcal{T}_{\omega_i}(\varphi)}{d\varphi}\right|_{\varphi=2\pi}$.

Since these curves ${}^d\mathcal{R}({}^d\varphi)$ and ${}^d\mathcal{I}({}^d\varphi)$ are periodic (period 2π), a Fourier series can be obtained to approximate them[3]:

$$\begin{aligned} \mathcal{R}(\varphi) &= A_{r0} + B_{r1}\sin\varphi + A_{r1}\cos\varphi + \\ &\quad + B_{r2}\sin 2\varphi + A_{r1}\cos 2\varphi + \ldots \\ \mathcal{I}(\varphi) &= A_{i0} + B_{i1}\sin\varphi + A_{i1}\cos\varphi + \\ &\quad + B_{i2}\sin 2\varphi + A_{i1}\cos 2\varphi + \ldots \end{aligned} \tag{5}$$

where

$$\begin{aligned} A_{r0} &= n\sum_{k=1}^{n} {}^d\mathcal{R}[k] \\ A_{rl} &= \frac{2}{n}\sum_{k=1}^{n} {}^d\mathcal{R}[k]\sin\left(l\,{}^d\varphi[k]\right) \\ B_{rl} &= \frac{2}{n}\sum_{k=1}^{n} {}^d\mathcal{R}[k]\cos\left(l\,{}^d\varphi[k]\right) \end{aligned} \tag{6}$$

A_{i0}, A_{il} and B_{il} can be computed analogously. Then the analytical and φ-parametric curve to describe the template is $\mathcal{T}_{\omega_i}(\varphi) = (\mathcal{R}(\varphi), \mathcal{I}(\varphi))$.

[3]Note that ${}^d\mathcal{R}$ is a array and $\mathcal{R}(\varphi)$ is an analytical function (the same for ${}^d\mathcal{I}$ and $\mathcal{I}(\varphi)$). ${}^d\varphi$ is the array that is associated to ${}^d\mathcal{R}$ and ${}^d\mathcal{I}$. φ is the continuous independent variable of $\mathcal{R}$ and $\mathcal{I}$.

Note that the discrete template is an array of complex numbers. If these numbers are taken in rectangular form to build an analytical template, a function $\mathcal{T}_{\omega_i}(\varphi) = (\mathcal{R}(\varphi), \mathcal{I}(\varphi))$ is obtained. When this function is evaluated, it provides points in the imag-real plane. Therefore, $(\mathcal{R}(\varphi), \mathcal{I}(\varphi))$ is the analytical template in rectangular form. If the discrete template is defined by means of complex numbers in polar form, the analytical template is obtained in polar form $\mathcal{T}_{\omega_i}(\varphi) = (\mathcal{M}(\varphi), \mathcal{A}(\varphi))$. The previous methodology describes how analytical templates in rectangular form are calculated. To do it in polar form a similar procedure is used.

B. Addition of analytical templates

The addition of two analytical templates $\mathcal{T}^1_{\omega_i}(\upsilon) = \left(\mathcal{R}^1_{\omega_i}(\upsilon), \mathcal{I}^1_{\omega_i}(\upsilon)\right)$ and $\mathcal{T}^2_{\omega_i}(\psi) = \left(\mathcal{R}^2_{\omega_i}(\psi), \mathcal{I}^2_{\omega_i}(\psi)\right)$ in rectangular form can be interpreted in this way: first, discretize $\upsilon = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$; then, obtain the curves $\mathcal{C}_1 = \mathcal{T}^1_{\omega_i}(\upsilon_1) + \mathcal{T}^2_{\omega_i}(\psi)$, $\mathcal{C}_2 = \mathcal{T}^1_{\omega_i}(\upsilon_2) + \mathcal{T}^2_{\omega_i}(\psi)$, $\ldots$, $\mathcal{C}_m = \mathcal{T}^1_{\omega_i}(\upsilon_m) + \mathcal{T}^2_{\omega_i}(\psi)$, that is the family $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$; finally, the envelope curve of this family is the required template. An example is shown in Fig. 4.

Every point on this envelope curve can be calculated from the intersection of $\mathcal{C}_h$ and $\mathcal{C}_{h+1}$ if they are infinitesimally close. Taking $\mathcal{T}^1_{\omega_i}$ and evaluating it by means of two infinitesimally close values: υ_1 and $\upsilon_2 = \upsilon_1 + \varepsilon_\upsilon$ ($\varepsilon_\upsilon \to 0$):

$$\begin{aligned} \mathcal{T}^1_{\omega_i}(\upsilon_1) &= p_{r1} + j\,p_{j1} \\ \mathcal{T}^1_{\omega_i}(\upsilon_1 + \varepsilon_\upsilon) &= p_{r2} + j\,p_{j2} \end{aligned}$$

Adding these points to $\mathcal{T}^2_{\omega_i}$ to obtain two infinitesimally close curves:

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{T}^2_{\omega_i}(\psi) + p_{r1} + j\,p_{j1} = \\ &= \left(\mathcal{R}^2_{\omega_i}(\psi) + p_{r1}, \mathcal{I}^2_{\omega_i}(\psi) + p_{j1}\right) \\ \mathcal{C}_2 &= \mathcal{T}^2_{\omega_i}(\psi) + p_{r2} + j\,p_{j2} = \\ &= \left(\mathcal{R}^2_{\omega_i}(\psi) + p_{r2}, \mathcal{I}^2_{\omega_i}(\psi) + p_{j2}\right) \end{aligned} \tag{7}$$

The solution of the next non-linear equation system gives the intersection points between $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\begin{cases} \mathcal{R}^2_{\omega_i}(\psi_1) + p_{r1} = \mathcal{R}^2_{\omega_i}(\psi_2) + p_{r2} \\ \mathcal{I}^2_{\omega_i}(\psi_1) + p_{j1} = \mathcal{I}^2_{\omega_i}(\psi_2) + p_{j2} \end{cases} \tag{8}$$

p_{r1} and p_{r2} are very close one to another (as well as p_{j1} and p_{j2}) because ε_υ tends to zero. Therefore, ψ_1 and $_2$ are also close. Performing $\psi_1 = \psi$ and $\psi_2 = \psi + \varepsilon_\psi$ (where $\varepsilon_\psi \to 0$):

$$\begin{cases} \mathcal{R}^2_{\omega_i}(\psi) + p_{r1} = \mathcal{R}^2_{\omega_i}(\psi) + \frac{d\,\mathcal{R}^2_{\omega_i}(\psi)}{d\psi}\varepsilon_\psi + p_{r2} \\ \mathcal{I}^2_{\omega_i}(\psi) + p_{j1} = \mathcal{I}^2_{\omega_i}(\psi) + \frac{d\,\mathcal{I}^2_{\omega_i}(\psi)}{d\psi}\varepsilon_\psi + p_{j2} \end{cases} \tag{9}$$

By removing ε_ψ, the following equation is obtained:

$$(p_{j1} - p_{j2})\frac{d\,\mathcal{R}^2_{\omega_i}(\psi)}{d\psi} = (p_{r1} - p_{r2})\frac{d\,\mathcal{I}^2_{\omega_i}(\psi)}{d\psi} \tag{10}$$

Dividing both terms by ε_υ and calculating the limit at $\varepsilon_\upsilon \to 0$:

$$\frac{d\,\mathcal{I}^1_{\omega_i}(\upsilon)}{d\upsilon}\frac{d\,\mathcal{R}^2_{\omega_i}(\psi)}{d\psi} = \frac{d\,\mathcal{R}^1_{\omega_i}(\upsilon)}{d\upsilon}\frac{d\,\mathcal{I}^2_{\omega_i}(\psi)}{d\psi} \tag{11}$$

and

$$\frac{\frac{d\,\mathcal{I}^1_{\omega_i}(\upsilon)}{d\upsilon}}{\frac{d\,\mathcal{R}^1_{\omega_i}(\upsilon)}{d\upsilon}} = \frac{\frac{d\,\mathcal{I}^2_{\omega_i}(\psi)}{d\psi}}{\frac{d\,\mathcal{R}^2_{\omega_i}(\psi)}{d\psi}} \tag{12}$$

Since $\frac{d\,\mathcal{T}_{\omega_i}(\varphi)}{d\varphi} = \frac{d\,\mathcal{R}_{\omega_i}(\varphi)}{d\varphi} + \frac{d\,\mathcal{I}_{\omega_i}(\varphi)}{d\varphi}$, the following conclusion is drawn:

$$\arg \mathcal{T}^{1'}_{\omega_i}(\upsilon) = \arg \mathcal{T}^{2'}_{\omega_i}(\psi) \tag{13}$$

where $'$ denotes the derivative[4]. This result is similar to the one shown in [15]. Using logarithm properties, these authors perform the template multiplication. When logarithms are used, accuracy mistakes can arise because a large variation in the template involves a small variation in the template logarithm, particularly when the template values are large. Another alternative is proposed below.

C. Multiplication of analytical templates

Take two templates $\mathcal{T}^1_{\omega_i}(\upsilon) = \big(\mathcal{A}^1_{\omega_i}(\upsilon), \mathcal{M}^1_{\omega_i}(\upsilon)\big)$ and $\mathcal{T}^2_{\omega_i}(\psi) = \big(\mathcal{A}^2_{\omega_i}(\psi), \mathcal{M}^2_{\omega_i}(\psi)\big)$, expressed in polar form ($\mathcal{A}$ denotes the argument and $\mathcal{M}$ the module). Discretize $\upsilon = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$ to calculate the curves $\mathcal{C}_1 = \mathcal{T}^1_{\omega_i}(\upsilon_1) \times \mathcal{T}^2_{\omega_i}$, $\mathcal{C}_2 = \mathcal{T}^1_{\omega_i}(\upsilon_2) \times \mathcal{T}^2_{\omega_i}$, …, $\mathcal{C}_m = \mathcal{T}^1_{\omega_i}(\upsilon_m) \times \mathcal{T}^2_{\omega_i}$, giving a family $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$, whose envelope curve is the final template contour. The intersection of two infinitesimally close curves $\mathcal{C}_h$ and $\mathcal{C}_{h+1}$ gives the points of the desired template.

$\mathcal{T}^1_{\omega_i}$ is evaluated at υ_1 and $\upsilon_2 = \upsilon_1 + \varepsilon_\upsilon$ ($\varepsilon_\upsilon \to 0$):

$$\begin{aligned} \mathcal{T}^1_{\omega_i}(\upsilon_1) &= k_1 \angle \theta_1 \\ \mathcal{T}^1_{\omega_i}(\upsilon_1 + \varepsilon_\upsilon) &= k_2 \angle \theta_2 \end{aligned}$$

Multiplying these points by $\mathcal{T}^2_{\omega_i}$, two infinitesimally close curves are yielded:

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{T}^2_{\omega_i}(\psi) \times k_1 \angle \theta_1 = \\ &= \big(\mathcal{A}^2_{\omega_i}(\psi) + \theta_1, \mathcal{M}^2_{\omega_i}(\psi) k_1\big) \\ \mathcal{C}_2 &= \mathcal{T}^2_{\omega_i}(\psi) \times k_2 \angle \theta_2 = \\ &= \big(\mathcal{A}^2_{\omega_i}(\psi) + \theta_2, \mathcal{M}^2_{\omega_i}(\psi) k_2\big) \end{aligned} \tag{14}$$

The intersection points are calculated as the solution of a non-linear equation system:

$$\left\{ \begin{array}{l} k_1\, \mathcal{M}^2_{\omega_i}(\psi_1) = k_2\, \mathcal{M}^2_{\omega_i}(\psi_2) \\ \mathcal{A}^2_{\omega_i}(\psi_1) + \theta_1 = \mathcal{A}^2_{\omega_i}(\psi_2) + \theta_2 \end{array} \right. \tag{15}$$

Since ε_υ tends to zero, ψ_1 and ψ_2 are close. Then $\psi_1 =$ and $\psi_2 = \psi + \varepsilon_\psi$ (where $\varepsilon_\psi \to 0$):

$$\left\{ \begin{array}{l} k_1\, \mathcal{M}^2_{\omega_i}(\psi) = k_2\, \mathcal{M}^2_{\omega_i}(\psi) + k_2 \frac{d\,\mathcal{M}^2_{\omega_i}(\psi)}{d\psi} \varepsilon_\psi \\ \mathcal{A}^2_{\omega_i}(\psi) + \theta_1 = \mathcal{A}^2_{\omega_i}(\psi) + \frac{d\,\mathcal{A}^2_{\omega_i}(\psi)}{d\psi} \varepsilon_\psi + \theta_2 \end{array} \right. \tag{16}$$

Removing ε_ψ gives:

$$\frac{k_1 - k_2}{k_2} \frac{d\,\mathcal{A}^2_{\omega_i}(\psi)}{d\psi} \mathcal{M}^2_{\omega_i}(\psi) = (\theta_1 - \theta_2) \frac{d\,\mathcal{M}^2_{\omega_i}(\psi)}{d\psi} \tag{17}$$

Dividing both terms by ε_υ and making the limit at $\varepsilon_\upsilon \to 0$:

$$\mathcal{M}^2_{\omega_i}(\psi)\, \mathcal{A}^{2'}_{\omega_i}(\psi) \mathcal{M}^{1'}_{\omega_i}(\upsilon) = \mathcal{M}^1_{\omega_i}(\upsilon)\, \mathcal{A}^{1'}_{\omega_i}(\upsilon) \mathcal{M}^{2'}_{\omega_i}(\psi) \tag{18}$$

where $'$ denotes the derivative. This expression can also be given using the rectangular form of the analytical template:

$$\begin{aligned} &\frac{\mathcal{R}^1_{\omega_i}(\upsilon)\, \mathcal{R}^{1'}_{\omega_i}(\upsilon) + \mathcal{I}^1_{\omega_i}(\upsilon)\, \mathcal{I}^{1'}_{\omega_i}(\upsilon)}{\mathcal{R}^1_{\omega_i}(\upsilon)\, \mathcal{I}^{1'}_{\omega_i}(\upsilon) - \mathcal{I}^1_{\omega_i}(\upsilon)\, \mathcal{R}^{1'}_{\omega_i}(\upsilon)} = \\ &\qquad = \frac{\mathcal{R}^2_{\omega_i}(\quad)\, \mathcal{R}^{2'}_{\omega_i}(\psi) + \mathcal{I}^2_{\omega_i}(\psi)\, \mathcal{I}^{2'}_{\omega_i}(\psi)}{\mathcal{R}^2_{\omega_i}(\psi)\, \mathcal{I}^{2'}_{\omega_i}(\psi) - \mathcal{I}^2_{\omega_i}(\psi)\, \mathcal{R}^{2'}_{\omega_i}(\psi)} \end{aligned} \tag{19}$$

D. The algorithm

Following, it is included the algorithm for obtaining the final template contour when two sub-templates are operated by $\circledcirc$:

1) Obtain the analytical templates $\mathcal{T}^1_{\omega_i}(\upsilon)$ and $\mathcal{T}^1_{\omega_i}(\psi)$.
2) For $\Psi = \{\psi_1, \psi_2, \ldots, \psi_m\}$ perform steps 3 and 4.
3) Take a value of $\psi = \psi_k \in \Psi$ and obtain the value of $\upsilon = \upsilon_k$ that solve (13) or (18) depending on the operation.
4) A point on the final template contour is $\mathcal{T}^1_{\omega_i}(\upsilon_k) \circ \mathcal{T}^2_{\omega_i}(\psi_k)$.

This algorithm has to be applied repeatedly to perform the operations between the sub-templates in which the initial plant was split.

III. REMARKS

To perform addition and multiplication operations between templates, the derivative of $\mathcal{T}_{\omega_i}(\varphi)$ must exist. This is guaranteed by using Fourier series this is guaranteed but it is convenient to make the corners of the discrete template round to obtain smooth derivatives.

The operation must be performed between two Jordan templates[5] or between a Jordan template and an open template (with only one uncertain parameter). To operate between two open templates, the algorithms of [9]–[13] can be used.

When (13) or (18) are solved and the templates are convex only two intersection points are found. If a template is open and the other one is closed both points belong to the final template contour. If both templates are closed, one solution does not give a contour point and must be eliminated.

[4]If $\arg \mathcal{T}_{\omega_i}$ is denoted by $\mathcal{A}_{\omega_i}$, note that $\mathcal{A}'_{\omega_i} \neq \arg \mathcal{T}'_{\omega_i}$.

[5]A Jordan template is a contour template that is defined by a closed and simple curve.

When one or both templates are non-convex several intersection points can be obtained, but only one[6] of them gives a point on the final template contour; the rest must be eliminated.

Equations (13) and (18) are transcendental and must be solved numerically. Addition, subtraction and multiplication operations can be between two Fourier series. Therefore, it is only necessary to obtain the zeros of continuous and differentiable series. Then, optimal algorithms can be used to solve it.

IV. Example

To illustrate the method, consider a model of a direct current motor driving a load. The block diagram is:

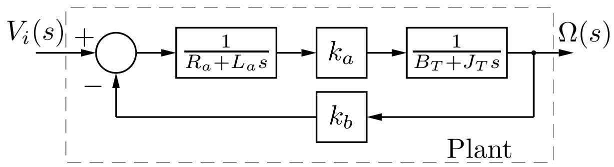

Fig. 6. Block diagram of a DC motor

The parameters present the following uncertainty: $R_a \in [1, 3.9]$, $L_a \in [1.4\,10^{-3}, 6.4\,10^{-3}]$, $k_a \in [1.4, 2.6]$, $k_b \in [1.5, 2.9]$, $J \in [4\,10^{-5}, 10^{-3}]$ and $B \in [10^{-5}, 3\,10^{-3}]$.

The model of the plant can be written as

$$\mathcal{P}(j\omega) = \frac{1}{\frac{(R_a + j\,L_a\omega)(B_T + j\,J_T\omega)}{ka} + kb} \qquad (20)$$

which can be split into four sub-plants: $P_1 = R_a + j\,L_a\omega$, $P_2 = \frac{1}{k_a}$, $P_3 = B_T + j\,J_T\omega$ and $P_4 = k_b$.

To calculate the template at $\omega = 100$ rad/s, the sub-templates $\mathcal{T}^1_{100}(\varphi_1)$, $\mathcal{T}^2_{100}(\varphi_2)$, $\mathcal{T}^3_{100}(\varphi_3)$ and $\mathcal{T}^4_{100}(\varphi_4)$ are obtained as shown in Fig. 7.

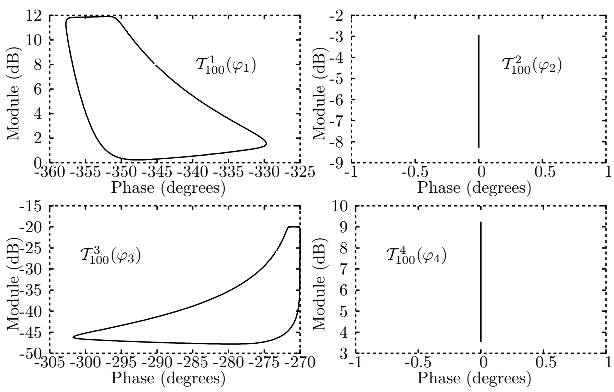

Fig. 7. Sub-templates of the DC motor

The next step is to perform operations between them. Figure 8 shows the template compositions. The final template $\mathcal{T}_{100}$ is obtained behind operating $1 \oslash \mathcal{T}^{1234}_{100}$; then $\mathcal{T}_{100}$ is the inverse[7] template of $\mathcal{T}^{1234}_{100}$.

[6]Or two of them when the operation is performed between open and closed templates.

[7]Note that the set of templates does not have inverse element, so $\mathcal{T}^{1234}_{100} \oslash \mathcal{T}^{1234}_{100}$ is not 1 but another template.

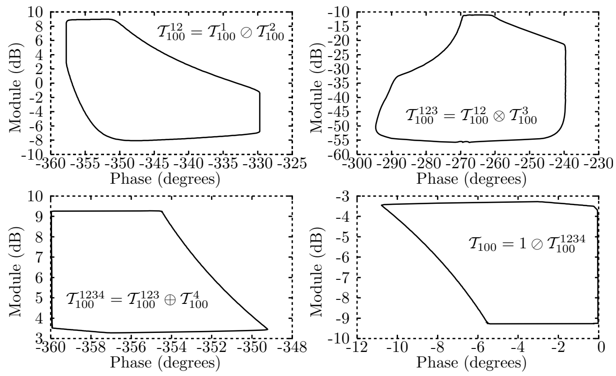

Fig. 8. Resulting templates when operations are performed

For the sake of clarity Fig. 9 includes the final template computed using a classical grid method (equidistant gridding [12]), which matches the result obtained with the proposed methodology; see Fig. 8.d).

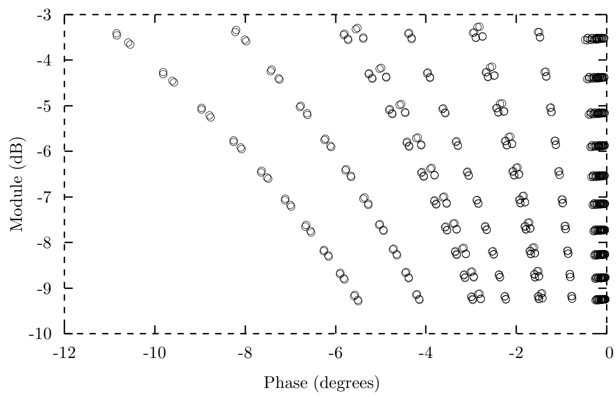

Fig. 9. Template at 100 rad/s using an equidistant grid method

V. Conclusions

A method to compute QFT templates has been presented. The method expands the one shown in [15], on this occasion performing multiplication between templates without logarithms. The method used to address the problem provides a formulation where computational time increases in arithmetic progression with the number of uncertain parameters instead of the geometric progression of the grid method.

The formulation has been designed to work with analytical templates. It allows the designer to take the whole uncertainty of the plant. It also opens a new math for the set of 'templates', where it is possible to define addition, subtraction, multiplication and division, as well as the commutative property. Addition has neutral element '0' (a template defined by one only point with a value '0'). Multiplication has neutral element '1'. There is no inverse element in addition or multiplication either.

Analytical templates have been defined by means of Fourier series. This allows to operations to be performed between templates in a simple way because the multiplication or addition of Fourier series produces other Fourier series. In the end, the problem is reduced to calculating the zeros of a series.

Looking at (13) and (18) it is possible to obtain very simple methods to perform operations between templates when one of them only suffers variation in its real or imaginary part, i.e. $\mathcal{T}_{\omega_i} \oplus (a + j\,\omega_i)$, where $\mathcal{T}_{\omega_i}$ is a closed template and 'a' is the only uncertain parameter of the second template.

ACKNOWLEDGMENTS

The authors gratefully appreciate the support given by the Spanish 'Comisión Interministerial de Ciencia y Tecnología (CICYT)' under grant DPI'2006-15522-C02-01 and by La Rioja Government under grant ANGI 2004/13.

REFERENCES

[1] I. M. Horowitz, *Synthesis of feedback systems.* Academic Press, 1963.

[2] I. M. Horowitz and M. Sidi, *Synthesis of feedback systems with large plant ignorance for prescribed time-domain tolerances.* Int. J. Control, 16, pp. 287-309, 1972.

[3] I. M. Horowitz, *Optimum loop transfer function in single-loop minimum-phase feedback systems.* Int. J. Control, Vol. 18 (1), 97-113, 1973.

[4] I. M. Horowitz, *Survey of Quantitative Feedback Theory (QFT).* Int. J. Control, 53, No. 2, pp. 255-291, 1991.

[5] I. M. Horowitz, *Quantitative Feedback Design Theory (QFT)* QFT Pub., 660 South Monaco Parkway, Denver, Colorado, 80224-1229, 1993.

[6] C. H. Houpis, S. J. Rasmussen and M. García-Sanz, *Quantitative Feedback Theory, Fundamentals and Applications* 2nd edition, Taylor and Francis, Florida, USA, 2006.

[7] O. Yaniv *Quantitative Feedback Design of Linear and Non-Linear Control Systems.* Kluwer Academic Publishers: Dordrecht, MA, USA, 1999.

[8] M. Fu, *Computing the frequency response of linear systems with parametric perturbations* Systems and Control Letters, Vol. 15, pp. 45-52, 1990.

[9] M. García-Sanz and P. Vital, *Efficient Computation of the frequency representation of uncertain systems.* 4th International symposium on quantitative feedback theory and robust frequency domain methods. 117-126, 1999.

[10] J. Cervera, A. Baños and I. M. Horowitz, *Computation of siso general plant templates.* 5th International symposium on quantitative feedback theory and robust frequency domain methods. pp. 247-254, 2001.

[11] D. J. Ballance and G. Hughes, *A survey of template generation methods for Quantitative Feedback Theory.* UKACC International conference on control '96. pp. 172-174, 1996.

[12] B. Cohen, M. Nordin and P. Gutman, *Recursive grid methods to compute value sets for transfer functions with parametric uncertainty.* Proceedings of the American control conference, Seattle, 1996.

[13] P. Gutman, M. Nordin and B. Cohen, *Recursive grid methods to compute value sets and Horowitz-Sidi bounds.* Int. J. of Robust and Nonlinear Control, Vol 17, pp. 155-171, 2006.

[14] E. Boje, *Finding nonconvex hulls of QFT templates.* Transactions of the ASME. Vol 122, pp. 230-232, 2000.

[15] A. Rantzer and P. Gutman, *Algorithm for addition and multiplication of value sets of uncertain transfer functions.* Proceedings of the 30th conference on decision and control, Brighton, 1991.

[16] P. Gutman, C. Baril and L. Neumann, *An algorithm for computing value sets of uncertain transfer functions in factored real form.* IEEE transactions on automatic control. Vol 39, pp. 1268-1273, 1994.

[17] D. J. Ballance and W. Chen, *Symbolic computation in value sets of plants with uncertain parameters.* UKACC International conference on control '98. pp. 1322-1327, 1998.

[18] P.S.V. Nataraj and G. Sardar, *Template generation for continuous transfer functions using interval analysis.* Automatica, vol. 36, pp. 111-119, 2000.

[19] P.S.V. Nataraj, *Interval QFT: a Mathematical and Computational Enhancement of QFT.* Int. J. of Robust and Nonlinear Control, Vol. 12, No. 4, pp. 385-402, 2002.

[20] V.G. Boltianski, *La envolvente* Editorial MIR Moscú, 1977.